Solutions

1. Find a function f which satisfies the conditions $f'(x) = 3e^x + x$, and $f(0) = 4$.

Solution: We want an antiderivative of $f'(x)$:

$$\int (3e^x + x)\, dx = 3\int e^x\, dx + \int x\, dx = 3e^x + \frac{1}{2}x^2 + C.$$

Now we need $f(0) = 4$, so $3e^0 + \frac{1}{2}0^2 + C = 4$, hence $C = 1$. So $f(x) = 3e^x + \frac{1}{2}x^2 + 1$.

2. A particle moves with velocity $v = 4 - t$. Find the total distance traveled by the particle after seven seconds.

Solution: The total distance traveled is given by $\int_0^7 |v(t)|\,dt$. We need to find when $v(t)$ is positive and negative: $4 - t = 0$ when $t = 4$; when $t < 4$, $v(t) > 0$, and when $t > 4$, $v(t) < 0$. So the total distance traveled is given by

$$\int_0^4 (4-t)\,dt + \int_4^7 -(4-t)\,dt = \left[4t - \frac{t^2}{2}\right]_0^4 + \left[\frac{t^2}{2} - 4t\right]_4^7 = (16-8)-(0-0)+\left(\frac{49}{2} - 28\right)-(8-16) = \frac{17}{2}.$$

Note that you cannot just take $\int_0^7 v(t)\,dt$, as this would calculate the total displacement rather than distance traveled.

3. Evaluate

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\left(\left(\frac{k}{n}\right)^2-5\left(\frac{k}{n}\right)\right).$$

Solution:

$$\begin{aligned}
\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\left(\left(\frac{k}{n}\right)^2-5\left(\frac{k}{n}\right)\right) &= \lim_{n\to\infty}\frac{1}{n}\left(\sum_{k=1}^{n}\frac{k^2}{n^2}-5\sum_{k=1}^{n}\frac{k}{n}\right)\\
&= \lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{n^2}\sum_{k=1}^{n}k^2-\frac{5}{n}\sum_{k=1}^{n}k\right)\\
&= \lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{n^2}\frac{n(n+1)(2n+1)}{6}-\frac{5}{n}\frac{n(n+1)}{2}\right)\\
&= \lim_{n\to\infty}\left(\frac{1(1+1/n)(2+1/n)}{6}-\frac{5(1+1/n)}{2}\right)\\
&= \frac{2}{6}-\frac{5}{2}\\
&= \frac{-13}{6}.
\end{aligned}$$

4. Evaluate the following integrals.

(a) $\displaystyle\int_0^4 \left(\sqrt{x} + 3x\right)\ dx$

Solution:

$$= \left[\frac{2}{3}x^{3/2} + \frac{3}{2}x^2\right]_0^4 = \frac{2}{3}\cdot 8 + \frac{3}{2}\cdot 16 = \frac{88}{3}$$

(b) $\displaystyle\int 4\frac{\sin\theta}{\cos^2\theta}\ d\theta$

Solution: Let $u = \cos\theta$, so $du = -\sin\theta\, d\theta$. Then the integral becomes

$$-4\int \frac{1}{u^2}\, du = -4(-u^{-1}) + C = \frac{4}{\cos\theta} + C$$

(c) $\displaystyle\int_0^2 \frac{e^x}{1+e^{2x}}\ dx$

Solution: Let $u = e^x$, so $du = e^x\, dx$. The integral becomes

$$\int_1^{e^2} \frac{1}{1+u^2}\, du = [\arctan u]_1^{e^2} = \arctan(e^2) - \arctan(1) = \arctan(e^2) - \frac{\pi}{4}.$$

(d) $\displaystyle\int \frac{x+1}{(x^2+2x-1)^2}\ dx$

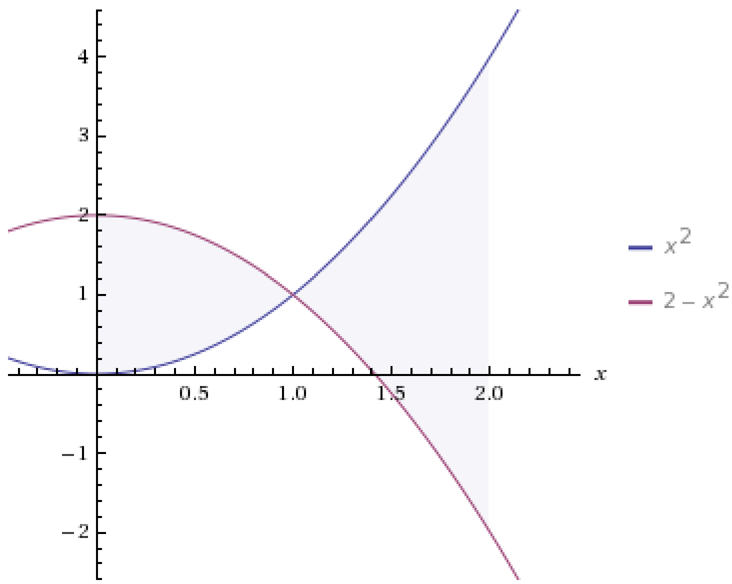
Solution: Let $u = x^2 + 2x - 1$, so $du = (2x+2)\, dx = 2(x+1)\, dx$. The integral becomes

$$\frac{1}{2}\int \frac{1}{u^2}\, du = -\frac{1}{2u} + C = -\frac{1}{2(x^2+2x-1)} + C.$$

5. Find the area between the graphs of the curves $y = x^2$ and $y = 2 - x^2$ on the interval $0 \leq x \leq 2$.

Solution: A quick sketch shows that these curves cross at $(1, 1)$. The area between the curves is

$$\begin{aligned}\int_0^1 [(2 - x^2) - (x^2)]\, dx + \int_1^2 [(x^2) - (2 - x^2)]\, dx &= \left[2x - \frac{2}{3}x^3\right]_0^1 + \left[\frac{2}{3}x^3 - 2x\right]_1^2 \\ &= \left(2 - \frac{2}{3}\right) - (0 - 0) + \left(\frac{16}{3} - 4\right) - \left(\frac{2}{3} - 2\right) \\ &= 4.\end{aligned}$$



6. (a) State the Mean Value Theorem.

Solution: Suppose f is a function that is continuous on $[a, b]$ and differentiable on (a, b). Then there is a $c \in (a, b)$ where $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

(b) Prove that the equation

$$x^4 + 6x^2 - 1 = 0$$

has exactly two real solutions.

Solution: Let $f(x) = x^4 + 6x^2 - 1$. Then $f(0) = -1$ and $f(1) = 6$, so since f is continuous on $[0, 1]$ (it is a polynomial), the Intermediate Value Theorem implies that there is a root in $(0, 1)$. Now f is an even function, so it also has a root in $(-1, 0)$. We now need to show that it has no other roots. Since f is even, we can show this by showing that it doesn't have two *positive* roots.

Suppose that it did have two positive roots, $a, b > 0$ with $f(a) = 0 = f(b)$. Then since f is continuous on $[a, b]$ and differentiable on (a, b) (again, it's a polynomial), Rolle's Theorem (or Mean Value Theorem) implies that there is a $c \in (a, b)$ where $f'(c) = 0$. Notice in particular that since $a > 0$, so is $c > 0$.

But direct calculation shows $f'(x) = 4x^3 + 6x = 2x(2x^2 + 3) > 0$ for $x > 0$. So there is no positive x-value where the derivative is zero, hence there cannot be two positive roots.

7. Let $\mathcal{R}$ be the region bounded by the curve $y = \sqrt{1-x^2}$ and the x-axis.

(a) Compute the volume of the solid obtained by rotating $\mathcal{R}$ about the x-axis.

Solution: Notice that this curve is the top half of a circle of radius one centered at the origin: $y^2 = 1 - x^2$, so $x^2 + y^2 = 1$. Rotating the region about the x-axis just gives a sphere of radius one, whose volume is $\frac{4}{3}\pi$. (You could also compute this with disks, but why would you?)

(b) Compute the volume of the solid obtained by rotating $\mathcal{R}$ about the line $x = -2$.

Solution: Using cylindrical shells is easiest here:

$$\begin{aligned}\int_{-1}^{1} 2\pi(x-(-2))(\sqrt{1-x^2})\,dx &= 2\pi\int_{-1}^{1}(x+2)\sqrt{1-x^2}\,dx \\ &= \pi\int_{-1}^{1} 2x\sqrt{1-x^2}\,dx + 4\pi\int_{-1}^{1}\sqrt{1-x^2}\,dx.\end{aligned}$$

The first integral is zero since we are integrating an odd function over a symmetric interval. Recognize the second integral as representing the area of the half-circle. We get

$$\begin{aligned}&= 4\pi\left(\frac{1}{2}\pi(1)^2\right) \\ &= 2\pi^2.\end{aligned}$$

8. Find the average value of the function $f(x) = x^3 \cos x$ on the interval $[-\pi, \pi]$.

Solution:

$$f_{avg} = \frac{1}{\pi - (-\pi)} \int_{-\pi}^{\pi} x^3 \cos x \, dx.$$

But the function in question is odd ($f(-x) = (-x)^3 \cos(-x) = -x^3 \cos x = -f(x)$), so the integral is zero, and hence so is the average value.