Workshop 23 November 29, 2011

1. The parametric equations for the involute of the circle of radius r (the unwinding string from last time) are

$$x = r\cos t + rt\sin t, \qquad y = r\sin t - rt\cos t,$$

where $t \geq 0$ is the angle from the x-axis to the point of tangency of the string to the circle. Use this to find polar coordinates for the involute. (The formula for r in terms of t is reasonably nice, but putting this in terms of θ won't be so pretty.)

2. We can think of polar coordinates as a function from $\mathbb{R}^2 \to \mathbb{R}^2$. For each pair of numbers $(r, \theta) \in \mathbb{R}^2$ we get as output the point $(x, y) \in \mathbb{R}^2$ where $x = r\cos\theta$ and $y = r\sin\theta$.

   (a) One oddity of this mapping is that it is not 1-to-1: for example the points $(r, \theta) = (3, \pi)$ and $(r, \theta) = (3, 3\pi)$ get mapped to the same point in (x, y)-space. Of course, there are even more such pairs (r, θ); describe all of them.

   (b) Another problem is the preimage of the origin. What points in (r, θ)-space map to the point $(x, y) = (0, 0)$?

   (c) Keeping the above issues in mind, find all points of intersection between the two polar curves $r = 1 + \sin\theta$ and $r = 1 - 2\cos\theta$. (Hint: you won't be able to solve all of the equations that come up explicitly; just simplify them a bit.)

3. Polar coordinates are great for describing some shapes, not so great for others. Find functions r of θ whose polar plots give rise to the following curves. Compare these to the Cartesian equations.

   (a) A line with slope $\sqrt{3}$ through the origin.

   (b) A line with slope m through the origin.

   (c) The horizontal line $y = 3$.

   (d) The circle with radius ρ centered at the origin.

   (e) The circle with radius ρ centered at $(x, y) = (5, 0)$.

   (f) The parabola $y = x^2$.

4. Show that the curves $r = a\sin\theta$ and $r = a\cos\theta$ intersect at right angles.

5. Find the length of the curve given parametrically by $x = e^t + e^{-t}$, $y = 5 - 2t$, $t \in [0, 3]$.

6. Find the length of the loop of the curve $x = 3t - t^3$, $y = 3t^2$.