Workshop 18 November 1, 2011

1. What is the Maclaurin series for $p(x) = x^{32} - \pi x^4 + 3124x^3 - 42x^2 - \ln 2^{\pi \sin(1)} x - 1$?

2. Find a Taylor series for $q(x) = x^\pi$ centered at 1. (Why can't I ask for a Maclaurin series?)

3. Use the following steps to prove that $\cosh x \le e^{x^2/2}$ for all x.

   (a) Recall that $\cosh x = \frac{1}{2}(e^x + e^{-x})$. Use the Maclaurin series expansion for e^x to get the Maclaurin series expansion for $\cosh x$.

   (b) Double-check your previous answer by computing the derivatives of $\cosh x$ at zero.

   (c) By ignoring the odd terms in $(2n)!$ and factoring out many twos, show that $(2n)! > 2^n n!$. Use this to give an upper bound on the series expansion of $\cosh x$.

   (d) How does your answer compare to the Maclaurin series expansion of $e^{x^2/2}$?

4. Find the first few terms of the product of series

$$\left(1 + x + x^2 + x^3 + \cdots\right) \cdot \left(1 - x + x^2 - x^3 + \cdots\right),$$

   then guess what the resulting series is. Recognizing the above series as Maclaurin series, what function is this product equal to?

5. *Convergence Issues*

   We've been a bit optimistic in our dealings with Taylor series so far. It is possible to start with a function $f(x)$, find its Taylor series $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$, and with it some interval of convergence, say $[a-R, a+R]$. So we have two functions, f and the function to which the series converges, but these *may not be the same*, even when $x \in [a-R, a+R]$. Here's an example. Let

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

   Compute the first few derivatives of $f(x)$ at $x = 0$. (Remark: pretend our shortcut rules work at $x = 0$, and evaluate all divisions by zero as though they were limits. [If we were being fully calc-1 rigorous, this problem would take a lot of work, since the piecewise definition nullifies our usual shortcut rules at zero; however, you can check (with much wailing and gnashing of teeth) that the definition of $f(0) = 0$ makes f infinitely differentiable.])

Make a guess at (or even better, give an argument that proves) what the nth derivative of f is. So what is the Maclaurin series for f? What is its interval of convergence?