Workshop 17 October 27, 2011

1. Find closed forms for each of the following; also find their intervals of convergence:

   (a) $\displaystyle\sum_{n=0}^{\infty} x^n$

   (b) $\displaystyle\sum_{n=0}^{\infty} nx^{n-1}$

   (c) $\displaystyle\sum_{n=0}^{\infty} n(n-1)x^{n-2}$

   (d) $\displaystyle\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$

   (e) $\displaystyle\sum_{n=0}^{\infty} x^{2n}$

   Evaluate the following:

   (i) $\displaystyle\sum_{n=0}^{\infty} \frac{n}{2^n}$

   (ii) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n2^n}$

2. Define the function $G(x) = \sum_{n=0}^{\infty} f_n x^n$, where $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$.

   (a) What is the radius of convergence for $G(x)$? (Recall that $\lim_{n\to\infty} f_{n+1}/f_n = \varphi = (1+\sqrt{5})/2$.)

   (b) What is a closed-form (i.e. no Sigma) expression for $G(x)$? (Hint: consider $xG(x)$ and $x^2G(x)$.)

   (c) What are the values of the zeroth, first, second, and third derivatives of this function when evaluated at $x = 0$?

3. Using the (soon to be) known equation $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x,

   (a) find a power series for e^{-x^2};

(b) find a power series for an antiderivative of e^{-x^2}.

Remember, we can't write down a closed form expression for $\int e^{-x^2}\, dx$!

4. Suppose we have a power series $\sum_{n=0}^{\infty} a_n(x-c)^n$. Find the first several derivatives of this function, evaluating each at $x = c$. What pattern emerges? Now, if we're given a function $f(x)$ and a point c in its domain, suppose we want to try to write down a power series centered at c representing f. At the very least, the derivatives of f and of the power series should match up. Use this fact to say something about the coefficients in the power series. Does this tell you everything about the series, or is there still information we can build into it?