Workshop 16 October 25, 2011

1. Let's get the value of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ another way today.

   (a) Start by writing $\frac{1}{1+x}$ as a power series centered about 0.

   (b) Now integrate the function. It turns out that you can integrate power series in the natural way, by treating them just as (very long) polynomials. Don't forget the constant though!

   (c) You should now have a power series representation for $\ln(x+1)$ (you already solved for the constant, right?). Plug in $x = 1$ to see what you get.

   (d) By the way, what were the intervals of convergence of the original series and the integrated series?

2. *Commutativity is Weird in Infinite Sums*

   Actually, I should say that commutativity is weird in conditionally convergent series. Consider the series

   $$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \cdots + \frac{1}{2k+1} - \frac{1}{4k+2} - \frac{1}{4k+4} + \cdots .$$

   Verify that this series is a rearrangement of the alternating harmonic series (i.e., check that all the terms show up here eventually). Now, simplifying the first two terms from each chunk of three terms, find the value of the series.

   (If you're being careful, you might object to the simplifications you made; after all, it's a bit like the associativity we demonstrated didn't work a few worksheets ago. If you're worried that we've lied above, consider the partial sums of the above series and compare them to the partial sums of the simplified series. Why does this justify the simplification?)

   (It turns out that you can make the above series converge to whatever your favorite number is (if it's a real number or $\pm\infty$) with an appropriate rearrangement!)

3. What is a power series anyway? Is $\sum_{n=0}^{\infty} \frac{1}{x^n}$ a power series? For what values of x does this series converge?

4. What is the interval of convergence for $\sum_{n=1}^{\infty} n! \cdot x^n$?

5. For each of the following intervals, either find a power series with that interval as its interval of convergence, or justify why there is none.

   $$(-1, 1] \qquad \{5\} \qquad [0, \infty) \qquad (-\infty, \infty) \qquad [-1, 1]$$

6. Define the function $G(x) = \sum_{n=0}^{\infty} f_n x^n$, where $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$.

   (a) What is the radius of convergence for $G(x)$? (Recall that $\lim_{n\to\infty} f_{n+1}/f_n = \varphi = (1+\sqrt{5})/2$.)

   (b) What is a closed-form (i.e. no Sigma) expression for $G(x)$? (Hint: consider $xG(x)$ and $x^2 G(x)$.)

   (c) What are the values of the zeroth, first, second, and third derivatives of this function when evaluated at $x = 0$?

7. If $f(x) = \frac{x}{1+x^4}$, what is $f^{(17)}(0)$?