Workshop 14 October 18, 2011

1. You know the p-test for series now:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \begin{cases} \text{converges for } p > 1; \\ \text{diverges for } p \leq 1. \end{cases}$$

(Remember that this is very much like the corresponding integral p-test.) What if we leave the series looking like $\sum \frac{1}{n}$, but throw away some of the integers n? Specifically, prove that

(a) the sum of the reciprocals of the even integers diverges (use some algebra);

(b) the sum of the reciprocals of the odd integers diverges (use a comparison and the previous problem);

(c) the sum of the reciprocals of the multiples of 42 diverges (more algebra);

(d) the sum of the reciprocals of the Fibonacci numbers converges (recall that we saw $\lim_{n\to\infty} f_{n+1}/f_n = \varphi$, the golden ratio.)

Do you think that the sum of the reciprocals of the primes converges or diverges? (You probably cannot prove this one.)

2. The Limit Comparison Test should be fairly intuitive. If $\lim_{n\to\infty} a_n/b_n = L$ and $0 < L < \infty$, then eventually the terms behave like multiples of each other (formally speaking, they are within arbitrarily small distances of those multiples). Then, since constant nonzero multiples can be factored out of the sum, the two series $\sum a_n$ and $\sum b_n$ either both converge or both diverge. Use this intuitive notion to extend the Limit Comparison test to when $L = 0$ and $L = \infty$. (Hint: you cannot get a statement like "they both ..."; instead, if you know that $\sum a_n$ converges, can you say whether $\sum b_n$ converges? What if you know $\sum a_n$ diverges? What if you know about $\sum b_n$?)

3. Use the Limit Comparison Test to determine whether $\displaystyle\sum_{k=1}^{\infty} \sin\left(\frac{\pi}{k}\right)$ converges or diverges. (Hint: You need an expression that is close in value to our terms, but is already known to either converge or diverge. Use the linearization, a.k.a. the tangent line, to $y = \sin(x)$ near $x = 0$ as an estimate for the sine. Then when using the Limit Comparison Test, you'll need to remember an important limit from Calc1.)

4. For the following series, try to use the alternating series test to determine convergence or divergence. In this problem, check ALL the hypotheses (even if one fails, in which case usually you wouldn't bother checking the others). If the alternating series test doesn't apply, try another method to determine convergence/divergence.

   (a) $\sum_{n=1}^{\infty} \frac{1}{n}$

   (b) $\sum_{n=1}^{\infty} \frac{\sin(n)}{n}$ (just say why the alternating series test doesn't apply)

   (c) $\sum_{n=1}^{\infty} a_n$, where $a_n = \begin{cases} \frac{1}{n} & \text{if } n \text{ is even;} \\ -\frac{1}{n^2} & \text{if } n \text{ is odd.} \end{cases}$

   (d) $\sum_{n=0}^{\infty} (-1)^n$

   (e) $\sum_{n=1}^{\infty} (-1)^n \left(\frac{n+1}{n} \right)$

   (f) $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$

   (g) $\sum_{n=2}^{\infty} (-1)^n \frac{1}{n \ln(n) \ln(\ln(n))}$