Workshop 13 October 11, 2011

1. *Estimating Series* You've been told that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ (Euler proved this back in 1741). We'll try to get close to this value (but won't prove the actual value).

   (a) First give rough bounds for the series by evaluating a related integral (that is, apply the integral test to show that the series converges, but keep track of the value of the integral(s); how are these values related to the value of the series?). Sketch a picture to justify that your bounds are correct.

   (b) Now determine how many terms of the series you need to add to guarantee that the result is correct to within 10^{-3}. If your calculator is capable, find this partial sum. (If not, you can just ask me.) Check the value of $\pi^2/6$ to make sure these values are really within 10^{-3}.

   (c) If you just add 20 terms, what bound does an integral give you on the error?

2. You can actually evaluate the following series. Do so.

   (a) $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$

   (b) $\sum_{n=5}^{\infty} \frac{1}{n^2+2n}$

   (c) $\sum_{n=42}^{\infty} \frac{1}{3^{2n+5}}$

3. Determine whether the following converge or diverge. If you can, give their value.

   (a) $\int_3^{\infty} e^{-2x+3}\,dx$

   (b) $\int_1^{\infty} x^{-1/2}\,dx$

   (c) $\int_1^{\infty} \frac{dx}{(x-\pi)^4}$

4. Determine whether the following converge or diverge. If you can, give their value. Be careful to check any assumptions!

   (a) $\sum_{n=3}^{\infty} e^{-2n+3}$

   (b) $\sum_{n=1}^{\infty} n^{-1/2}$

   (c) $\sum_{n=1}^{\infty} \frac{1}{(n-\pi)^4}$

5. For numbers whose decimal expansions never terminate, we use series (through limits) if we want to talk technically about their values.

(a) Find the value of the rational number that we represent by the decimal $0.14222222\ldots$. (Hint: separate the nonrepeating and repeating parts.)

(b) Find the value of the rational number that we represent by the decimal $0.12121212\ldots$. (Hint: group together the blocks of repeated digits. If you're feeling adventurous, do the same for the decimal $0.142857142857142857\ldots$.)

(c) Find the value of the rational number that we represent by the decimal $0.9999999\ldots$.

(d) The above series were eventually geometric (presumably that's how you found their values). What about irrational numbers? Their series aren't geometric; prove anyway that any decimal expansion actually converges (if it didn't, our number system wouldn't be very nice!). Hint: use an appropriate theorem!

6. Set up integrals that give the surface area of the following solids.

   (a) The "ellipsoid" formed by rotating the ellipse $x^2 + 9y^2 = 36$ about the x-axis.

   (b) The surface formed by rotating the right half of the unit circle about the line $x = -3$.

   (c) A (right circular) cone whose base circle has radius r and whose height is h.

7. Set up an integral representing the arc length of the curve $y = \ln x$ between $(1, 0)$ and $(e, 1)$. (You can actually evaluate this integral with a bit of simplification, a u-sub, and partial fraction decomposition; do it if you want the practice.)

8. *using the arc length differential for hydrostatic force*

   Whenever we computed hydrostatic force before, the surfaces were always lamina, i.e. flat plates. We can use the same method for arbitrary surfaces, but we need to be a little more careful about the area of a strip. But you've already dealt with area of strips in the context of surfaces of revolution via the arc length differential.

   Suppose we have a spherical submarine with radius 2m submerged so that the center of the sphere is 10m below the surface.

   (a) We divide as usual the surface into thin strips at roughly constant depth. Sketch the situation together with one such strip. Make sure to mark your axes/ruler!

   (b) Compute the area of such a small strip (in terms of y).

   (c) Compute the pressure on this small strip.

   (d) Thus compute the force on the small strip.

   (e) Now add up all these forces, and limit the sum into an integral. Thus compute the total force exerted on the outside of the sphere.

9. Fix some real number α. Consider the sequence $b_n = \alpha^{\left(\alpha^{\cdot^{\cdot^{\alpha}}}\right)}$, where the tower has a total of n copies of α. So $b_1 = \alpha$, $b_2 = \alpha^\alpha$, and in general $b_n = \alpha^{b_{n-1}}$.

   (a) Suppose the limit of this sequence exists, and consider the recursive formula just given. Use this to find an equation that relates α and the limit L.

   (b) What value of α will give you a limit of $L = 2$?

   (c) What value of α will give you a limit of $L = 4$?

   (d) Simplify your previous two answers if necessary, then ask yourself what's going on. Why do your answers naively not make sense, and what went wrong?