Workshop 12 October 6, 2011

1. Show that for all n large enough,

$$\frac{1}{n} > \frac{1}{n \cdot \ln n} > \frac{1}{n \cdot \ln n \cdot \ln(\ln n)} > \frac{1}{n \cdot \ln n \cdot \ln(\ln n) \cdot \ln(\ln(\ln n))}.$$

   You know that the largest of the above ($1/n$) forms a divergent series, i.e.

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges.}$$

   What about the next one down? I.e., determine whether

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

   converges or diverges. What about the last two?

2. Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Graph the three functions

$$f(x) = \frac{1}{x^2} \qquad g(x) = f(\lfloor x \rfloor) \qquad h(x) = f(\lceil x \rceil)$$

   say for $1 \le x \le 4$. Using these graphs and simple geometric arguments, what can you say about the values of the following?

$$\int_2^{\infty} f(x)\,dx \qquad \int_1^{\infty} f(x)\,dx \qquad \int_1^{\infty} g(x)\,dx \qquad \int_1^{\infty} h(x)\,dx \qquad \sum_{n=1}^{\infty} \frac{1}{n^2} \qquad \sum_{n=2}^{\infty} \frac{1}{n^2}$$

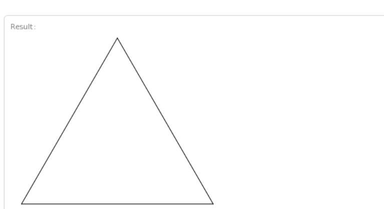
3. You showed last time that $\lim_{n\to\infty}(\ln(n+1) - \ln(n)) = 0$. Determine whether the series

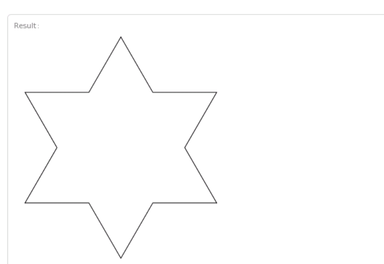
$$\sum_{n=1}^{\infty} (\ln(n+1) - \ln(n))$$

   converges or diverges. If it converges, can you find its value?

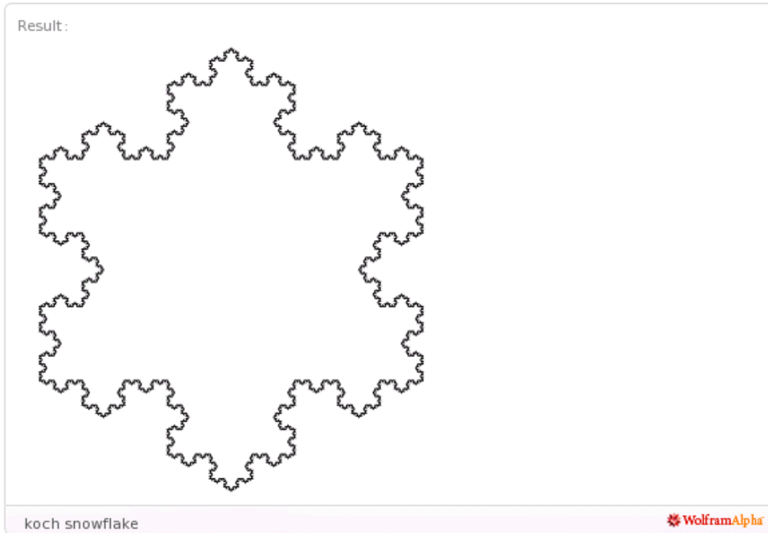
4. *More Fractals!*

   The *Koch snowflake* is another fractal (like the Sierpinski carpet you dealt with earlier). You start with an equilateral triangle, let's say its sides have length 1. Now stick new equilateral triangles to every edge, each with side length 1/3 of the original. These two steps are shown below.





Now attach even more equilateral triangles onto every edge (including all those new ones!), each with side length 1/3 of the previous triangles. Rinse and repeat forever. Here's what we have after the 5th iteration:



Using appropriate sequences and/or series, find the length of the outside curve and the area contained within. (Hint: you should get weirdness.)

5. Two players take turns flipping a fair coin; whoever flips the first heads wins. What is the probability that the first player wins? (Hint: find the probability that player 1 wins in the first toss, the second toss, etc. and add these up. Note that the probability that player 1 wins in the third toss is the probability of getting T, T, H)

6. Can you make the game in 5 fair by using a weighted coin? That is, if the coin comes up heads with probability p ($0 \leq p \leq 1$), what value(s) of p make the probability that the first player wins equal to 1/2?

7. You know the p-test for series now:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \begin{cases} \text{converges for } p > 1; \\ \text{diverges for } p \leq 1. \end{cases}$$

(Remember that this is very much like the corresponding integral p-test.) Problem 1 gives you some series that diverge but even more slowly than the harmonic series, but we relied on making the denominators bigger by factors of $\ln n$. What if we leave the series looking like $\sum \frac{1}{n}$, but throw away some of the integers n? Specifically, do you think that the sum of the reciprocals of the even integers converges? Odd integers? Multiples of 42? Primes? Prove as many of your answers as you can.