Workshop 11 October 4, 2011

1. A lot of the sequences you see you'll want to compare to functions of real variables. For example, if you're considering the sequence $a_n = \frac{n^4}{e^n}$, you might investigate the function $f(x) = \frac{x^4}{e^x}$. How are these different, aside from changing the symbols? Complete the following implications (hint: some of the answers are "????"; it may be helpful to sketch some graphs).

   (a) $\lim_{n\to\infty} a_n = L \implies \lim_{x\to\infty} f(x)$ ____________

   (b) $\lim_{n\to\infty} a_n$ doesn't exist $\implies \lim_{x\to\infty} f(x)$ ____________

   (c) $\lim_{x\to\infty} f(x) = L \implies \lim_{n\to\infty} a_n$ ____________

   (d) $\lim_{x\to\infty} f(x)$ doesn't exist $\implies \lim_{n\to\infty} a_n$ ____________

2. Use the previous problem to evaluate the following.

   (a) $\lim_{n\to\infty} \frac{\ln n}{n}$

   (b) $\lim_{n\to\infty} \frac{e^k}{k^{42}}$

   (c) $\lim_{n\to\infty} \ln(n+1) - \ln(n)$

3. Sometimes that approach doesn't work so well, for instance when the "function" $f(x)$ doesn't make sense. Be a little clever to find the following.

   (a) $\lim_{n\to\infty} 1 + (-1)^n$

   (b) $\lim_{n\to\infty} \frac{(-1)^n n}{n^2+1}$

   (c) $\lim_{n\to\infty} \frac{n^{42}}{n!}$

   (d) $\lim_{n\to\infty} \frac{(42)^n}{n!}$

   (e) $\lim_{n\to\infty} \frac{(-1)^n n}{n+1}$

4. *Associativity is Weird in Infinite Sums*

   What do you think $1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \cdots$ is? If we're allowed to put parentheses wherever we want (addition is associative), then perhaps it's

   $$(1-1) + (1-1) + (1-1) + \cdots = 0 + 0 + 0 + \cdots = 0,$$

   but also we have

   $$1 + (-1+1) + (-1+1) + (-1+1) + \cdots = 1 + 0 + 0 + 0 + \cdots = 1.$$

   What's going on here?

5. The *Fibonacci sequence* is defined by $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. (Sometimes people use different starting points.)

   (a) Write down the first few terms of this sequence. Does this sequence have a limit?

   (b) Suppose we have a sequence b_n, and define another sequence $c_n = b_{n-1}$ (so we just shift the terms of the sequence). Why is $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n$?

   (c) Define a new sequence $a_n = f_{n+1}/f_n$, $n \geq 1$. Write down the first few terms of this sequence. Use the definition of the Fibonacci sequence to show that $a_{n+1} = 1 + \dfrac{1}{a_n}$.

   (d) Use 5b and 5c to find the limit $\lim_{n\to\infty} a_n$ (assuming it exists; if you really want an explanation for why it exists, chat with me).