Workshop 7 September 20, 2011

1. What does it mean geometrically for an improper integral to converge? Diverge?
2. Does the integral $\int_1^\infty \frac{\sin x + \pi}{\sqrt{x}}\,dx$ converge or diverge? Can you tell whether $\int_1^\infty \frac{\sin x}{\sqrt{x}}\,dx$ converges?
3. Suppose you know that the function $f(x)$ is continuous on $[0, \infty)$, and that $|f(x)| < 1/x^2$ for $x \geq 10^{10}$. Can you conclude whether $\int_0^\infty f(x)\,dx$ converges or diverges?
4. Suppose one of your classmates makes the following calculation:

$$\int_{-\infty}^{\infty} x\,dx = \lim_{t\to\infty} \int_{-t}^{t} x\,dx = \lim_{t\to\infty} \frac{1}{2}x^2\Big|_{-t}^{t} = \frac{1}{2}\lim_{t\to\infty}(t^2 - t^2) = 0.$$

   What's gone wrong here?

Improper Integrals are Important!

1. The *Laplace transform* is an invaluable tool in many applications. It is an operation that transforms one function into another, but some properties of the resulting functions are easier to deal with. The Laplace transform of the function $f(t)$ is

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}\,dt.$$

   (a) Compute the Laplace transformations of the functions 1, t, t^2, t^3, and in general t^n for any positive integer n.

   (b) Compute the Laplace transform of the function $f(t) = e^{at}$. (Note: for what s does the integral converge?)

   (c) Prove that $\mathcal{L}\{f'(t)\} = s \cdot \mathcal{L}\{f(t)\} - f(0)$ by computing $\mathcal{L}\{f(t)\}$ with integration by parts.

2. Sometimes we want to discuss probabilities when the set of outcomes isn't finite. For instance, perhaps we want to talk about the expected length of time one waits in line at the DMV, instead of talking about the probability of a coin flip. To do this, we use a continuous *probability density function* $f(t)$, and the probability of, say, waiting between 3 and 4 hours is given by $\int_3^4 f(t)\,dt$. Perhaps at the local DMV this function is given by

$$f(t) = te^{-t}.$$

   (a) What is the probability that you wait some amount of time between 0 hours and forever? (What should it be? Compute it as $\int_0^\infty f(t)\,dt$.)

   (b) The *expected value*, roughly speaking the time you expect to wait, is given by $\int_0^\infty t \cdot f(t)\,dt$. Find the expected waiting time.

   (c) Repeat the first two parts, this time with the function $f(t) = 1/t^2$.