Worksheet 1 August 25, 2011

1. We know several rules for differentiation. Give the rule for how to differentiate the following types of functions. Do you know a rule for antidifferentiation based on them?
   (a) sum of two functions
   (b) product of two functions
   (c) constant multiple of a function (this is a special case of 1b)
   (d) composition of two functions

2. Compute $\displaystyle\int \frac{x}{x^2+2x+2}\,dx$.

3. Compute $\displaystyle\lim_{x\to 0}\frac{e^x-1-x-x^2/2}{x^3}$. For now, use L'Hopital; later, you'll be able to do this using McLaurin series.

4. Compute $\displaystyle\int \sin^{123}(x)\cos^3(x)\,dx$.

5. *The Wallis Product*
   Let $I_n = \int_0^{\pi/2}\sin^n(x)\,dx$ for positive integers n.
   (a) Use integration by parts to prove the following reduction formula for $n \geq 2$ an integer:
   $$I_n = \frac{n-1}{n} I_{n-2}.$$
   (b) Justify the following facts about the value of I_k for odd and even k using the reduction formula:
   $$I_{2n+1} = \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdot\dots\cdot\frac{2n}{2n+1},$$
   $$I_{2n} = \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\dots\cdot\frac{2n-1}{2n}\cdot\frac{\pi}{2}.$$
   (If you've heard of mathematical induction, that's a convenient way to formally prove these statements. I don't care about the formality here though, so just give a good explanation.)
   (c) Use the second formula to show that
   $$\frac{I_{2n+2}}{I_{2n}} = \frac{2n+1}{2n+2}.$$
   (d) Explain, considering appropriate graphs, why $I_{2n+2} \leq I_{2n+1} \leq I_{2n}$ for any n.
   (e) From the previous two parts, show that
   $$\frac{2n+1}{2n+2} \leq \frac{I_{2n+1}}{I_{2n}} \leq 1$$
   for all n. Then prove that
   $$\lim_{n\to\infty}\frac{I_{2n+1}}{I_{2n}} = 1.$$

(f) To pull all this together, show that

$$\lim_{n\to\infty}\left(\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\dots\cdot\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\right)=\frac{\pi}{2}.$$

If we loosen the notation a bit, this is the cute infinite product

$$\frac{\pi}{2}=\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\dots\,.$$

6. This one's a long one, but not really difficult. Remember I said that you can't write down an antiderivative of e^{x^2} in terms of the functions you know? Well, I needed to say that the formula can't be *finite* (and also, I'm assuming you don't know of the "error function"...). We can get an infinite expression for this function by using integration by parts.

   (a) So we want to compute $\int e^{x^2}\,dx$. Apply integration by parts. (What choices do you have here?)

   (b) You should end up with an expression that has one ordinary looking term and one integral. Try to evaluate this integral with integration by parts. (What choices do you have now? Are any of the choices obviously not going to help?)

   (c) Now you should have two integral-free terms and one integral. You can expand this integral by parts again.

   (d) Continue as far as you like, or until you see a pattern arising (you shouldn't simplify multiplications, but do distribute across parentheses after each step).