Solutions

1. Find the sum of the series

$$\frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n+1}}{16^n(2n+1)!}.$$

A. 0

B. $\frac{1}{2}$

C. $\frac{1}{\sqrt{2}}$

D. $\frac{\sqrt{3}}{2}$

E. The series does not converge.

Solution: We have that

$$\begin{aligned}\frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n+1}}{16^n(2n+1)!} &= \sum_{n=0}^{\infty}(-1)^n\frac{\left(\frac{\pi}{4}\right)^{2n+1}}{(2n+1)!}\\ &= \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}.\end{aligned}$$

2. Find the sum of the series

$$\sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}2^n.$$

A. $\sqrt{3}$

B. $\sqrt{2}$

C. $\frac{1}{\sqrt{3}}$

D. $\frac{1}{\sqrt{2}}$

E. The series does not converge.

Solution: We have that the binomial series

$$\sqrt{1+x} = \sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}x^n$$

has radius of convergence $R = 1$. Therefore $x = 2$ does not lie in its interval of convergence, and the sum in question diverges.

3. Which of the following is the Maclaurin series of $x \arctan\left(3x^2\right)$?

   A. $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{3^{2n+1}x^{4n+2}}{2n+1}$

   B. $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{3^{2n+1}x^{4n+3}}{2n+1}$

   C. $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{3^{4n+3}x^{4n+3}}{2n+1}$

   D. $\displaystyle\sum_{n=0}^{\infty}(-1)^{2n+1} \frac{3^{2n+1}x^{4n+3}}{2n+1}$

   E. $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{3^{2n+1}x^{2n+1}}{2n+1}$

Solution: The Maclaurin series for $\arctan\left(3x^2\right)$ is

$$\arctan\left(3x^2\right) = \sum_{n=0}^{\infty}(-1)^n \frac{(3x^2)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}(-1)^n \frac{3^{2n+1}x^{4n+2}}{2n+1},$$

so the Maclaurin series for $x\arctan\left(3x^2\right)$ is

$$x\arctan\left(3x^2\right) = \sum_{n=0}^{\infty}(-1)^n \frac{3^{2n+1}x^{4n+3}}{2n+1}.$$

4. Evaluate the limit

$$\lim_{x\to 0} \frac{\sin x - x + \frac{1}{6}x^3}{x^5}.$$

   A. 0

   B. $-\frac{1}{3!}$

   C. $\frac{1}{5!}$

   D. $-\frac{1}{7!}$

   E. $\frac{1}{9!}$

Solution: We have that $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$, so

$$\sin x - x + \frac{1}{6}x^3 = \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots,$$

and

$$\lim_{x\to 0} \frac{\sin x - x + \frac{1}{6}x^3}{x^5} = \lim_{x\to 0} \frac{\frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots}{x^5} = \lim_{x\to 0}\left(\frac{1}{5!} - \frac{1}{7!}x^2 + \cdots\right) = \frac{1}{5!}.$$

5. If the Maclaurin series for the function $f(x) = \sqrt{1+x}$ is used to approximate $\sqrt{1.1}$, what is the minimum number of terms required to achieve three decimal places of accuracy?

   A. One

   B. Two

   C. Three

   D. Four

   E. Five

Solution: The Maclaurin series for f is

$$\sqrt{1+x} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} x^n.$$

Using this series to approximate $\sqrt{1.1}$, we get

$$\sqrt{1.1} = \sqrt{1+10^{-1}} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} 10^{-n} = 1 + \frac{1}{2}10^{-1} - \frac{1}{8}10^{-2} + \frac{1}{16}10^{-3} - \cdots,$$

which is an alternating series (after the first term). Since $a_3 = \frac{1}{16}(10^{-3})$ is the first term to have absolute value smaller than 10^{-3}, the desired sum is $a_0 + a_1 + a_2$, i.e. three terms are required.

6. Do the following series converge or diverge?

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{4^n}{3^n + 5^n}$

Solution: We do a limit comparison with the series $\displaystyle\sum_{n=0}^{\infty} \left(\frac{4}{5}\right)^n$. We get that

$$\lim_{n\to\infty} \left(\frac{4^n}{3^n+5^n}\right)\left(\frac{5}{4}\right)^n = \lim_{n\to\infty} \frac{5^n}{3^n+5^n} = \lim_{n\to\infty} \frac{1}{\left(\frac{3}{5}\right)^n + 1} = 1.$$

Since the value of this limit is positive, and the series $\displaystyle\sum_{n=0}^{\infty} \left(\frac{4}{5}\right)^n$ is a convergent geometric series, by the limit comparison test, the original series converges.

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{\pi}{n}\right)$

Solution: For $n \geq 1$, $0 \leq \frac{\pi}{n} \leq \pi$, so $b_n = \sin\left(\frac{\pi}{n}\right) \geq 0$. Let $f(x) = \sin\left(\frac{\pi}{x}\right)$, so $f'(x) = -\frac{\pi}{x^2}\cos\left(\frac{\pi}{x}\right)$. We see that $f'(x) < 0$ when $0 \leq \frac{\pi}{x} < \frac{\pi}{2}$, which has solution $x > 2$. Therefore the sequence $\{b_n\}$ is decreasing for $n \geq 3$. Since

$$\lim_{n\to\infty} \sin\left(\frac{\pi}{n}\right) = \sin 0 = 0,$$

this series converges by the alternating series test.

7. Find the interval of convergence for each of the following series.

(a) $\sum_{n=1}^{\infty} n^n x^n$

Solution: Using the root test, we get that

$$\lim_{n\to\infty} \sqrt[n]{|n^n x^n|} = \lim_{n\to\infty} n\,|x| = \begin{cases} 0 & \text{if } x = 0, \\ \infty & \text{if } x \neq 0. \end{cases}$$

Except when $x = 0$, this limit is not less than one, so the series only converges at the center, when $x = 0$.

(b) $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$

Solution: Using the ratio test, we get that

$$\lim_{n\to\infty} \left| \left(\frac{(-3)^{n+1} x^{n+1}}{\sqrt{n+2}} \right) \left(\frac{\sqrt{n+1}}{(-3)^n x^n} \right) \right| = \lim_{n\to\infty} 3\,|x| \frac{\sqrt{n+1}}{\sqrt{n+2}} = 3\,|x|\,.$$

We therefore must solve the inequality $3\,|x| < 1$, which has solution $-\frac{1}{3} < x < \frac{1}{3}$. We now must check the endpoints $x = \pm\frac{1}{3}$.

When $x = \frac{1}{3}$, we get the series

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{\sqrt{n+1}}.$$

Clearly $b_n = \frac{1}{\sqrt{n+1}} \geq 0$, $\frac{1}{\sqrt{n+2}} \leq \frac{1}{\sqrt{n+1}}$, so $\{b_n\}$ is decreasing. Also,

$$\lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 0,$$

so by the alternating series test, this series converges.

When $x = -\frac{1}{3}$, we get the series

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}},$$

which diverges by the p-test.

The interval of convergence is therefore $\left(-\frac{1}{3}, \frac{1}{3}\right]$.

8. Consider the function

$$f(x) = \frac{e^{x^2}}{1-x^2}.$$

(a) Write the first three nonzero terms in the Maclaurin series of f.

Solution: We have that

$$\begin{aligned} f(x) = e^{x^2}\left(\frac{1}{1-x^2}\right) &= \left(1 + x^2 + \frac{1}{2}x^4 + \cdots\right)\left(1 + x^2 + x^4 + \cdots\right) \\ &= 1 + x^2 + x^4 + x^2 + x^4 + \frac{1}{2}x^4 + \cdots \\ &= 1 + 2x^2 + \frac{5}{2}x^4 + \cdots . \end{aligned}$$

(b) Compute $f^{(3)}(0)$ and $f^{(4)}(0)$.

Solution: The formula for the Maclaurin series for f is

$$f(x) = f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + \frac{1}{3!}f^{(3)}(0)x^3 + \frac{1}{4!}f^{(4)}(0)x^4 + \cdots .$$

Since the Maclaurin series for f computed in part (a) has no x^3 term, we get that $f^{(3)}(0) = 0$. Since the coefficient of the x^4 in the Maclaurin series for f computed in part (a) is $\frac{5}{2}$, we get that

$$f^{(4)}(0) = 4!\left(\frac{5}{2}\right) = 60.$$

9. Compute

$$\sum_{n=1}^{\infty} \frac{n}{2^n}.$$

Solution: For $|r| < 1$, we have

$$\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n.$$

Differentiating this equation, we get

$$\frac{1}{(1-r)^2} = \sum_{n=1}^{\infty} nr^{n-1},$$

so

$$\frac{r}{(1-r)^2} = \sum_{n=1}^{\infty} nr^n.$$

In particular, when $r = \frac{1}{2}$,

$$2 = \sum_{n=1}^{\infty} \frac{n}{2^n}.$$

10. Let f be a function that is infinitely differentiable at $x = a$.

(a) Define the Taylor series of f centered at $x = a$.

Solution: $T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$.

(b) Define the degree-N Taylor polynomial of f centered at $x = a$.

Solution: $T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n$.

(c) Define the Taylor remainder $R_N(x)$.

Solution: $R_N(x) = f(x) - T_N(x)$.

(d) What does Taylor's theorem say about $R_N(x)$?

Solution: $R_N(x) = \frac{f^{(N+1)}(z)}{(N+1)!}(x-a)^{N+1}$ for some z between a and x.