Solutions

1. True or false, $\displaystyle\sum_{n=2}^{\infty} \frac{n^{0.001}}{\ln n}$ converges.

   A. True **B. False**

> **Solution:** Let $f(x) = \dfrac{x^{0.001}}{\ln x}$. As $x \to \infty$, both the numerator and denominator tend to infinity, so we may apply L'Hôpital's rule to get
>
> $$\begin{aligned}\lim_{x\to\infty} \frac{x^{0.001}}{\ln x} &= \lim_{x\to\infty} \frac{0.001\ x^{-0.999}}{\frac{1}{x}} \\ &= \lim_{x\to\infty} 1.0001\ x^{0.001} \\ &= \infty.\end{aligned}$$
>
> Since $f(n) = \dfrac{n^{0.001}}{\ln n}$, we get that
>
> $$\lim_{n\to\infty} \frac{n^{0.001}}{\ln n} = \infty.$$
>
> Since the limit of the terms of this series is not zero, the series diverges.

2. Where is the integral

$$\int_0^{\pi} \tan\theta\ d\theta$$

improper?

   A. Nowhere
   B. $\frac{\pi}{4}$
   C. $\frac{\pi}{2}$
   D. $\frac{3\pi}{4}$
   E. π

> **Solution:** The interval of this integral is finite, so we find where the integrand, $\tan\theta$, is discontinuous. Since $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ is discontinuous when $\cos\theta = 0$, we see that the answer is $\theta = \frac{\pi}{2}$.

3. True or false, the integral test can be applied to the series

$$\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{2}\right)}{n^2}?$$

A. True **B. False**

Solution: We see that $\sin\left(\frac{n\pi}{2}\right) = -1$ infinitely many times, specifically, when $\frac{n\pi}{2} = \frac{3\pi}{2} + 2k\pi$. Since $\frac{1}{n^2} \geq 0$ for all n, the terms of this series are negative infinitely many times, so the integral test is not applicable, and the answer is false.

4. True or false, the series

$$\sum_{n=1}^{\infty} \frac{n+1}{n^3}$$

converges?

A. True B. False

Solution: Consider the function $f(x) = \dfrac{x+1}{x^3}$. When $x \geq 1$, $x+1 \geq 0$ and $x^3 > 0$, so $f(x) \geq 0$. Since $f(x) = x^{-2} + x^{-3}$, $f'(x) = -2x^{-3} - 3x^{-4} < 0$, for $x \geq 1$, so f is decreasing. Finally,

$$\lim_{x\to\infty} \frac{x+1}{x^3} = \lim_{x\to\infty} \left(\frac{1}{x^2} + \frac{1}{x^3}\right) = 0,$$

so the integral test is applicable. For $x \geq 1$,

$$f(x) = \frac{x+1}{x^3} \leq \frac{x+x}{x^3} = \frac{2}{x^2}.$$

Since

$$\int_1^{\infty} \frac{2}{x^2}\, dx$$

converges by the p-test, by the comparison test,

$$\int_1^{\infty} f(x)\, dx$$

converges. By the integral test, the series converges.

5. Which of the following integrals represents the area of the surface obtained by rotating the curve $y = \sin x$ for $0 \le x \le \pi$ about the x-axis?

   A. $\displaystyle\int_0^\pi \sin x\sqrt{1+\cos^2 x}\;dx$

   B. $\displaystyle 2\pi\int_0^\pi \sin x\sqrt{1+\sin^2 x}\;dx$

   C. $\displaystyle 2\pi\int_0^\pi \cos x\sqrt{1+\sin^2 x}\;dx$

   D. $\displaystyle 2\pi\int_0^\pi \sin x\sqrt{1+\cos^2 x}\;dx$

   E. $\displaystyle\int_0^\pi \sqrt{1+\cos^2 x}\;dx$

Solution: The equation for the area of a surface obtained by rotating a function of x on the interval $[0, \pi]$ is

$$A = 2\pi\int_0^\pi y\;ds,$$

where $ds = \sqrt{dx^2 + dy^2}$. Since $y = \sin x$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \cos x$, and

$$ds = \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\;dx = \sqrt{1+\cos^2 x}\;dx,$$

The area of this surface is therefore

$$2\pi\int_0^\pi \sin x\sqrt{1+\cos^2 x}\;dx.$$

6. Show that

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1} = \frac{3}{4}.$$

Solution: The partial fractions decomposition of the summand is

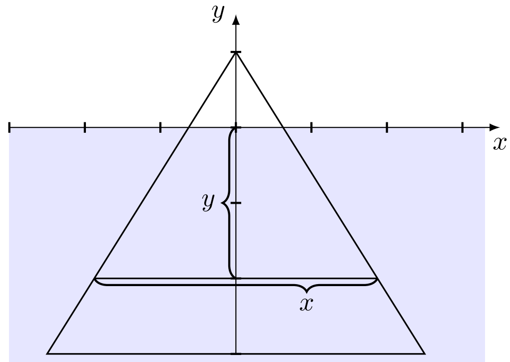
$$\frac{1}{n^2-1} = \frac{1}{2}\left(\frac{1}{n-1} - \frac{1}{n+1}\right).$$

We therefore get that

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{1}{n^2-1} &= \frac{1}{2}\sum_{n=2}^{\infty}\left(\frac{1}{n-1} - \frac{1}{n+1}\right) \\
&= \frac{1}{2}\left(\underbrace{\left(1 - \cancel{\frac{1}{3}}\right)}_{n=2} + \underbrace{\left(\frac{1}{2} - \cancel{\frac{1}{4}}\right)}_{n=3} + \underbrace{\left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right)}_{n=4} + \underbrace{\left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{6}}\right)}_{n=5} + \cdots\right) \\
&= \frac{1}{2}\left(1 + \frac{1}{2}\right) \\
&= \frac{3}{4}.
\end{aligned}$$

7. A triangular lamina that is 4 m tall and 5 m wide is partially submerged, base-first, in water. It reaches a maximum depth of 3 m. Carefully sketch a diagram of this situation with clearly drawn and labeled axes. Compute the hydrostatic force on one side of the lamina.

Solution: We have the following diagram, where the water line coincides with the x-axis.



The differential of the hydrostatic force is given by $dF = \rho g d\ dA$. From the diagram, we see that $d = -y$ and $dA = x\ dy$. By considering similar triangles, we see that

$$\frac{x}{5} = \frac{1-y}{4},$$

so $x = \frac{5}{4}(1-y)$, and $dF = \frac{5}{4}\rho g y(y-1)\ dy$. The hydrostatic force is therefore

$$\begin{aligned} F &= \frac{5}{4}\rho g \int_{-3}^{0} y(y-1)\ dy \\ &= \frac{5}{4}\rho g \left(\frac{1}{3}y^3 - \frac{1}{2}y^2\right)\Big|_{-3}^{0} \\ &= \frac{135}{8}\rho g. \end{aligned}$$

8. Determine if the following series converge or diverge. If a series converges, find its sum.

(a) $\frac{1}{\pi} + \frac{1}{\pi^2} + \frac{1}{\pi^3} + \frac{1}{\pi^4} + \cdots$

Solution: This series is geometric, with ratio $\frac{1}{\pi}$ and first term $\frac{1}{\pi}$. Since $\pi > 3$, $0 < \frac{1}{\pi} < \frac{1}{3} < 1$, so the series converges to the value

$$\frac{\frac{1}{\pi}}{1 - \frac{1}{\pi}} = \frac{1}{\pi\left(1 - \frac{1}{\pi}\right)} = \frac{1}{\pi - 1}.$$

(b) $\sum_{n=0}^{\infty} \frac{2^{2n+1}}{3^n}$

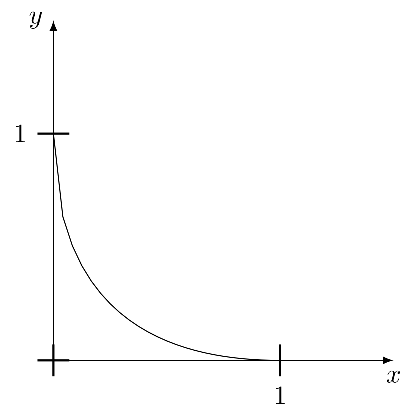
Solution: Since $2^{2n+1} = 2(4^n)$,

$$\sum_{n=0}^{\infty} \frac{2^{2n+1}}{3^n} = 2\sum_{n=0}^{\infty} \left(\frac{4}{3}\right)^n.$$

This series diverges, since $\frac{4}{3} > 1$.

9. Compute the centroid of the region bounded by the curve $\sqrt{x}+\sqrt{y}=1$, the x-axis, and the y-axis.

Solution: This region is shown the following diagram.



We first compute $\bar{x}$. Since $y=(1-\sqrt{x})^2$, and the x-intercept of this curve is $(1,0)$,

$$\begin{aligned} M_y &= \int_0^1 x(1-\sqrt{x})^2\ dx \\ &= \int_0^1 (x-2x^{\frac{3}{2}}+x^2)\ dx \\ &= \left(\frac{1}{2}x^2-\frac{4}{5}x^{\frac{5}{2}}+\frac{1}{3}x^3\right)\Big|_0^1 \\ &= \frac{1}{30}. \end{aligned}$$

The area of this region is

$$\begin{aligned} A &= \int_0^1 (1-\sqrt{x})^2\ dx \\ &= \int_0^1 (1-2\sqrt{x}+x)\ dx \\ &= \left(x-\frac{4}{3}x^{\frac{3}{2}}+\frac{1}{2}x^2\right)\Big|_0^1 \\ &= \frac{1}{6}. \end{aligned}$$

Therefore $\bar{x}=\frac{M_y}{A}=\frac{1}{5}$. Since the original equation is symmetric in x and y, we also get that $\bar{y}=\frac{1}{5}$. The centroid of this region is $(\frac{1}{5},\frac{1}{5})$.

10. Consider the series $\sum_{n=1}^{\infty} ne^{-n^2}$.

(a) Use the integral test to show that this series converges.

Solution: Since $e^{-n^2} \geq 0$ and $n \geq 1$, $ne^{-n^2} \geq 0$. Let $f(x) = xe^{-x^2}$, so f is continuous everywhere, and

$$f'(x) = x(-2x)e^{-x^2} + e^{-x^2} = (1 - 2x^2)e^{-x^2} < 0$$

for $x \geq 1$. Therefore f is decreasing. Writing f as a fraction, we see that as $x \to \infty$ both the numerator and denominator tend to infinity, so we may apply L'Hôpital's rule to get

$$\lim_{x\to\infty} \frac{x}{e^{x^2}} = \lim_{x\to\infty} \frac{1}{2xe^{x^2}} = 0.$$

We may therefore apply the integral test. Making the substitution $u = -x^2$, we compute

$$\begin{aligned}
\int_1^{\infty} xe^{-x^2}\, dx &= \lim_{R\to\infty} \int_1^R xe^{-x^2}\, dx \\
&= \lim_{R\to\infty} -\frac{1}{2} \int_{-1}^{-R^2} e^u\, du \\
&= \lim_{R\to\infty} -\frac{1}{2} \left(e^u\right)\Big|_{-1}^{-R^2} \\
&= \lim_{R\to\infty} \frac{1}{2} \left(e^{-1} - e^{-R^2}\right) \\
&= \frac{1}{2} e^{-1}.
\end{aligned}$$

Since this indefinite integral converges, the series converges by the integral test.

(b) How many terms are required to approximate the value of this series to within ten decimal places of accuracy?

Solution: If R_n represents the error of approximating the value of this series by the first n terms, we have that

$$R_n \leq \int_n^{\infty} xe^{-x^2}\, dx = \frac{1}{2} e^{-n},$$

by a computation very similar to the one in part (a). Five decimal places of accuracy represents an error smaller than 10^{-10}, so we solve the inequality

$$\frac{1}{2} e^{-n} < 10^{-10}.$$

This inequality has solution $n > \sqrt{\ln\left(\frac{1}{2} \cdot 10^{10}\right)}$. We therefore must use at least $\sqrt{\ln\left(\frac{1}{2} \cdot 10^{10}\right)} \approx 5$ terms to attain ten decimal places of accuracy.