Solutions

1. Compute $\int (x+1)e^x\,dx$.

Solution: We integrate by parts with

$$\begin{array}{cc} u = x+1 & dv = e^x\ dx \\ du = dx & v = e^x, \end{array}$$

and get

$$\int (x+1)e^x\ dx = (x+1)e^x - \int e^x\ dx = (x+1)e^x - e^x + C = xe^x + C. \tag{1}$$

2. Use integration by parts to evaluate $\int \sec^3 x\,dx$.

Solution: We integrate by parts with

$$\begin{array}{cc} u = \sec x & dv = \sec^2 x\ dx \\ du = \sec x \tan x\ dx & v = \tan x, \end{array}$$

and get

$$\int \sec^3 x\ dx = \sec x \tan x - \int \tan^2 x \sec x\ dx \tag{2}$$

$$= \sec x \tan x - \int (\sec^2 x - 1)\sec x\ dx \tag{3}$$

$$= \sec x \tan x - \int \sec^3 x\ dx + \int \sec x\ dx \tag{4}$$

$$= \sec x \tan x - \int \sec^3 x\ dx + \ln|\sec x + \tan x| + C. \tag{5}$$

Adding the original integral to both sides, we get

$$2\int \sec^3 x\ dx = \sec x \tan x + \ln|\sec x + \tan x| + C, \tag{6}$$

so

$$\int \sec^3 x\ dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C, \tag{7}$$

3. Compute $\int \sec^3 x \tan^3 x \, dx$.

Solution: We decide to make the substitution $u = \sec x$, and therefore must save a factor of $du = \sec x \tan x \; dx$. We get

$$\begin{aligned}
\int \sec^3 x \tan^3 x \; dx &= \int \sec^2 x \tan^2 x (\sec x \tan x) \; dx && (8)\\
&= \int \sec^2 x (\sec^2 x - 1)(\sec x \tan x) \; dx && (9)\\
&= \int (u^4 - u^2) \; du && (10)\\
&= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C && (11)\\
&= \frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C. && (12)
\end{aligned}$$

4. Compute $\int \sqrt{4 - x^2} \, dx$.

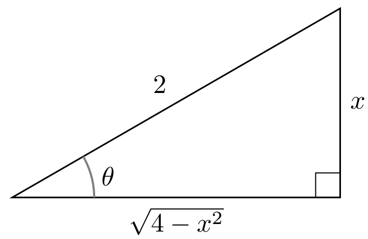
Solution: We make the trigonometric substitution $x = 2\sin\theta$, so $du = 2\cos\theta \; d\theta$, and we get

$$\begin{aligned}
\int \sqrt{4 - x^2} \; dx &= \int \sqrt{4 - 4\sin^2\theta}(2\cos\theta) \; d\theta && (13)\\
&= 4\int \sqrt{1 - \sin^2\theta}\cos\theta \; d\theta && (14)\\
&= 4\int \sqrt{\cos^2\theta}\cos\theta \; d\theta && (15)\\
&= 4\int |\cos\theta|\cos\theta \; d\theta. && (16)
\end{aligned}$$

When $\cos\theta \geq 0$, or $-\frac{\pi}{2} \leq 0 \leq \frac{\pi}{2}$, we get

$$\begin{aligned}
\int \sqrt{4 - x^2} \; dx &= 4\int |\cos\theta|\cos\theta \; d\theta && (17)\\
&= 4\int \cos^2\theta \; d\theta && (18)\\
&= 2\int (1 + \cos(2\theta)) \; d\theta && (19)\\
&= 2\left(\theta + \frac{1}{2}\sin(2\theta)\right) + C && (20)\\
&= 2\theta + 2\sin\theta\cos\theta + C. && (21)
\end{aligned}$$

Since $\sin\theta = \frac{x}{2}$ get the following triangle.



Therefore $\theta = \sin^{-1}\left(\frac{x}{2}\right)$ and $\cos\theta = \frac{1}{2}\sqrt{4-x^2}$, so

$$\int \sqrt{4-x^2}\ dx = 2\sin^{-1}\left(\frac{x}{2}\right) + \frac{1}{2}x\sqrt{4-x^2} + C. \tag{22}$$

5. Compute $\displaystyle\int \frac{x}{x^2-5x+6}\,dx$.

Solution: Since $x^2-5x+6=(x-2)(x-3)$, we compute the partial fractions decomposition of the integrand as

$$\frac{x}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}. \tag{23}$$

Therefore,

$$x = A(x-3) + B(x-2). \tag{24}$$

Letting $x=2$, we get $A=-2$, and letting $x=3$, we get $B=3$. Therefore

$$\begin{aligned} \int \frac{x}{x^2-5x+6}\,dx &= -2\int \frac{1}{x-2}\,dx + 3\int \frac{1}{x-3}\,dx & (25)\\ &= -2\ln|x-2| + 3\ln|x-3| + C. & (26) \end{aligned}$$

6. Compute $\displaystyle\int \frac{4x^2+x+5}{x^3+2x-3}\,dx$.

Solution: We first must factor the denominator. Since $x=1$ is a root of x^3+2x-3, we do polynomial long division and get

$$\begin{array}{r|l} & x^2 \quad + x + 3. \\ \hline x-1 & x^3 \qquad + 2x - 3 \\ & -x^3 + x^2 \\ \hline & x^2 + 2x \\ & -x^2 \ + x \\ \hline & 3x - 3 \\ & -3x + 3 \\ \hline & 0 \end{array}$$

Therefore $x^3+3x-3=(x-1)(x^2+x+3)$. We note that x^2+x+3 is irreducible, since its discriminant is $\Delta = 1-4(1)(3) = -11 < 0$. The partial fractions decomposition of the integrand is therefore

$$\frac{4x^2+x+5}{(x-1)(x^2+x+3)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+3}. \tag{27}$$

We get

$$\begin{aligned} 4x^2+x+5 &= A(x^2+x+3) + (Bx+C)(x-1) & (28)\\ &= Ax^2 + Ax + 3A + Bx^2 - Bx + Cx - C & (29)\\ &= (A+B)x^2 + (A-B+C)x + (3A-C). & (30) \end{aligned}$$

Since polynomials are equal when their like coefficients are equal, we get

$$A + B = 4, \tag{31}$$
$$A - B + C = 1, \tag{32}$$
$$\text{and } 3A - C = 5. \tag{33}$$

Solving this system of equations, we get $A = 2$, $B = 2$, and $C = 1$. Therefore

$$\int \frac{4x^2 + x + 5}{x^3 + 3x - 3}\ dx = 2\int \frac{1}{x-1}\ dx + \int \frac{2x+1}{x^2 + x + 3}\ dx \tag{34}$$
$$= 2\ln|x-1| + \int \frac{2x+1}{x^2+x+3}\ dx. \tag{35}$$

To evaluate the remaining integral, we substitute $u = x^2 + x + 3$, so $du = (2x + 1)\ dx$, and we get

$$\int \frac{2x+1}{x^2+x+3}\ dx = \int \frac{1}{u}\ du = \ln|u| + C = \ln\left|x^2 + x + 3\right| + C. \tag{36}$$

Combining equations 35 and 36, we get that

$$\int \frac{4x^2 + x + 5}{x^3 + 2x - 3}\ dx = 2\ln|x-1| + \ln\left|x^2 + x + 3\right| + C. \tag{37}$$

7. Compute $\displaystyle\int \frac{dx}{\sqrt{x}(2+\sqrt{x})^5}$.

Solution: We substitute $u = 2+\sqrt{x}$, so $du = \frac{1}{2\sqrt{x}}\,dx$, and we get

$$\int \frac{1}{\sqrt{x}(2+\sqrt{x})^5}\,dx = 2\int u^{-5}\,du = -\frac{1}{2}u^{-4} + C = -\frac{1}{2(2+\sqrt{x})^4} + C. \tag{38}$$

8. Compute $\displaystyle\int \frac{1+\sin t}{1-\sin t}\,dt$.

Solution: We multiply both the numerator and denominator of the integrand by $1+\sin t$ and get

$$\int \frac{1+\sin t}{1-\sin t}\,dx = \int \left(\frac{1+\sin t}{1-\sin t}\right)\left(\frac{1+\sin t}{1+\sin t}\right)\,dx \tag{39}$$

$$= \int \frac{(1+\sin t)^2}{1-\sin^2 t}\,dt \tag{40}$$

$$= \int \frac{1+2\sin t+\sin^2 t}{\cos^2 t}\,dt \tag{41}$$

$$= \int \frac{1}{\cos^2 t}\,dt + 2\int \left(\frac{1}{\cos t}\right)\left(\frac{\sin t}{\cos t}\right)\,dt + \int \frac{\sin^2 t}{\cos^2 t}\,dt \tag{42}$$

$$= \int \sec^2 t\,dt + 2\int \sec t\tan t\,dt + \int \tan^2 t\,dt \tag{43}$$

$$= \tan t + 2\sec t + \int (\sec^2 t - 1)\,dt \tag{44}$$

$$= 2\tan t + 2\sec t - t + C. \tag{45}$$