MATH 221 / Fall 2009
Disclaimer: This review was written by your TA, Ben Reiniger. The final exam will be written by Dr. Reznick. As such, these problems should not be taken as a complete representative of the actual final; rather, they are meant to be a supplement to your review of previous homework, quiz, and exam experiences. Problems marked with an asterisk are somehow different than what you've seen before: more challenging, more interesting, or otherwise.

SOLUTIONS

1. Evaluate:

i) $\lim_{x\to 4} \frac{(x-4)^2}{(x-4)^4} = \lim_{x\to 4} \frac{1}{(x-4)^2} = +\infty$ (DNE is sufficient).

ii) $\lim_{x\to 4} \frac{(x-4)^{253}}{(x-4)^{200}} = \lim_{x\to 4} (x-4)^{53} = 0.$

iii) $\lim_{x\to 4} \frac{8(x-4)^{3.14}}{(x-4)^{3.14}} = \lim_{x\to 4} 8 = 8.$

iv) $\lim_{x\to 4^+} \frac{\ln(x-4)}{x-4}$ is of the form $\frac{-\infty}{+0}$ so is $-\infty$ (again, "DNE" is sufficient).

v*) $\lim_{x\to 4^+} \frac{\ln(x-4)}{\tan\left(\frac{\pi x}{8}\right)}$ is of the form $\frac{-\infty}{-\infty}$ so is $\lim_{x\to 4^+} \frac{\frac{1}{x-4}}{\frac{\pi}{8}\sec^2\left(\frac{\pi x}{8}\right)} = \lim_{x\to 4^+} \frac{\cos^2\left(\frac{\pi x}{8}\right)}{\frac{\pi}{8}(x-4)}$

which is of the form $\frac{+0}{+0}$ so is $\lim_{x\to 4^+} \frac{\frac{2\pi}{8}\cos\left(\frac{\pi x}{8}\right)\sin\left(\frac{\pi x}{8}\right)}{\frac{\pi}{8}} = 0.$

vi) $\lim_{x\to 4} \frac{(x-5)^4}{(x-3)^{12}} = 1$

vii) $\lim_{x\to 0} x^2 \cos\frac{1}{x}$
(Hint: Squeeze!)
We have, since $x^2 \geq 0$ for all x, that $-x^2 \leq x^2 \cos\frac{1}{x} \leq x^2$ for all $x \neq 0$. Since $\lim_{x\to 0} -x^2 = 0 = \lim_{x\to 0} x^2$, the Squeeze Theorem gives that the limit of interest is also 0.

2. a) What is the formal definition of the statement "$\lim_{x\to 3} 3x - 4 = 5$"?
For any $\epsilon > 0$, there is a $\delta > 0$ such that whenever $0 < |x-3| < \delta$, we have that $|(3x-4)-5| < \epsilon$.

b) Prove the statement.
Let $\epsilon > 0$. Choose $\delta = \frac{\epsilon}{3}$. Then, if $0 < |x-3| < \delta$, we have that

$$|(3x-4)-5| = |3x-9| = 3|x-3| < 3\delta = \epsilon$$

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3. A squirrel runs along a fence with position at time t given by $x(t) = 2t^{3/2}$.

a) What is his average velocity between times $t = 1$ and $t = 4$?

$$v_{\text{ave}} = \frac{x(4) - x(1)}{4 - 1} = \frac{16 - 2}{3} = \frac{14}{3}.$$

b) What is his instantaneous velocity at time $t = 1$?

$$x'(t) = 3t^{1/2} \qquad x'(1) = 3.$$

c) What is the equation for the tangent line to the curve $x(t)$ at $t = 1$?
Since $x(1) = 2$, the tangent line is given by

$$\begin{aligned} L(t) - L(1) &= x'(1)(x - 1) \\ L(t) - 2 &= 3(x - 1) \\ L(t) &= 3x - 1. \end{aligned}$$

d) Use a linear approximation based at $t = 1$ to estimate the squirrel's position at time $t = 2$.
A linear approximation is found by the tangent line. So we estimate

$$x(2) \approx L(2) = 3(2) - 1 = 5.$$

4. The squirrel now leaps from his perch toward a passerby with ice cream. In an interesting happenstance of nature, the squirrel's height above his target (factoring in air resistance) is exactly $y = 16 - 2x^{3/2}$, where x is the horizontal distance traveled. What is the distance traveled by the flying rodent?

The distance traveled here is equal to the arclength. We need to know the endpoints to use. We start at $x = 0$ and go until the squirrel reaches his target, i.e. when $y = 0$. Then

$$0 = 16 - 2x^{3/2} \implies x^{3/2} = 8 \implies x = 4.$$

Then the distance is

$$\frac{1}{4-0}\int_0^4 \sqrt{1 + \left(-3x^{1/2}\right)^2}dx = \frac{1}{4}\int_0^4 \sqrt{1 + 9x}dx \qquad \text{take } u = 1 + 9x \text{ so } du = 9dx$$

$$= \frac{1}{36}\int_{u=1}^{35} \sqrt{u}du = \frac{1}{36}\frac{2}{3}\left[u^{3/2}\right]_1^{35} = \frac{1}{54}\left(35^{3/2} - 1\right).$$

5. a) Find a function $f(x)$ and a number a so that the following limit represents the definition of $f'(a)$:

$$\lim_{h \to 0} \frac{\sqrt{8 + h} - \sqrt{8}}{h}$$

$f(x) = \sqrt{x}$ and $a = 8$ works.

b) Use your answer to (a), along with your knowledge of the derivatives of elementary functions, to evaluate the limit.

$$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}} \qquad f'(8) = \frac{1}{2\sqrt{8}} = \frac{1}{4\sqrt{2}}.$$

c) Now evaluate the limit directly (hint: rationalize the numerator). Do NOT use L'Hopital's Rule.

$$\begin{aligned} \lim_{h \to 0} \frac{\sqrt{8 + h} - \sqrt{8}}{h} &= \lim_{h \to 0} \frac{\sqrt{8 + h} - \sqrt{8}}{h} \cdot \frac{\sqrt{8 + h} + \sqrt{8}}{\sqrt{8 + h} + \sqrt{8}} \\ &= \lim_{h \to 0} \frac{8 + h - 8}{h\left(\sqrt{8 + h} + \sqrt{8}\right)} = \lim_{h \to 0} \frac{1}{\sqrt{8 + h} + \sqrt{8}} = \frac{1}{2\sqrt{8}}. \end{aligned}$$

6. Consider the function $f(x) = x - \ln(x)$.
a) What is the domain of f? $\{x : x > 0\}$
b) Find intervals of increase, decrease, concave up, and concave down.

$$f'(x) = 1 - \frac{1}{x} \qquad\qquad f''(x) = \frac{1}{x^2}$$
$$f'(x) = 0 \Rightarrow x = 1. \qquad\qquad f''(x) \neq 0 \text{ for any } x$$

Since $f'(1/2) = -1/2 < 0$ and $f'(2) = 1/2 > 0$, f is decreasing on $(0, 1)$ and increasing on $(1, \infty)$.
Since $f''(x) > 0$ for all x, f is concave up on its domain, $(0, \infty)$.

c) Find and classify local extrema.
The only critical point is $x = 1$, where $f(1) = 1 - \ln(1) = 1$. Since f is decreasing then increasing, the point $(1, 1)$ is a local minimum.

d) Find inflection points.
There are no points where the second derivative is zero or undefined. Thus there are no inflection points.

e) Do we have a theorem which guarantees absolute extrema on $(0, 2]$? Find them if they exist.
No theorem applies here since we do not have a closed interval. However, there is an absolute minimum at $x = 1$. There is no absolute maximum, since as $x \to 0$ from the right, $f(x) \to +\infty$.

f) Do we have a theorem which guarantees absolute extrema on $\left[\frac{1}{e^3}, 3\right]$? Find them if they exist.
The Extreme Value Theorem does apply here (f is continuous on a closed interval). We consider the critical points and endpoints:

$$f\left(\frac{1}{e^3}\right) = \frac{1}{e^3} + 3 > 3 \qquad f(1) = 1 \qquad f(3) = 3 - \ln 3 < 3.$$

We conclude that $(1, 1)$ is the absolute minimum and $\left(\frac{1}{e^3},\ 3 + \frac{1}{e^3}\right)$ is the absolute maximum.

7. Where is the function

$$f(x) = \begin{cases} \sin x & \text{for } x < -\pi \\ -1 & \text{for } x = -\pi \\ x + \pi & \text{for } -\pi < x \leq 1 \\ -4x + \pi & \text{for } 1 < x \end{cases}$$

continuous? Justify your answers.
The function is continuous for all x except perhaps at $-\pi$ and 1 since the piecewise components are all continuous. At $x = -\pi$, we have

$$\lim_{x\to-\pi^-} f(x) = \sin(-\pi) = 0 \qquad \lim_{x\to-\pi^+} f(x) = -\pi + \pi = 0 \qquad f(-\pi) = -1$$

so the limit exists, but is not equal to the value, so f is not continuous here. And at $x = 1$,

$$\lim_{x\to1^-} f(x) = 1 + \pi \qquad \lim_{x\to1^+} f(x) = -4(1) + \pi = -4 + \pi \qquad f(1) = 1 + \pi$$

so the limit does not exist (left and right limits do not match), and f is not continuous here either.

8. Find the derivative of $y = x^{\tan^{-1} x}$.
Having x in both the base and the exponent is bad, so take logs:

$$\begin{aligned}
\ln(y) &= \tan^{-1}(x)\ln(x) \\
\frac{1}{y}y' &= \frac{1}{1+x^2}\ln(x) + \frac{1}{x}\tan^{-1}(x) \\
y' &= y\left(\frac{1}{1+x^2}\ln(x) + \frac{1}{x}\tan^{-1}(x)\right) \\
&= x^{\tan^{-1} x}\left(\frac{1}{1+x^2}\ln(x) + \frac{1}{x}\tan^{-1}(x)\right)
\end{aligned}$$

9. Suppose the number of yeast cells in a suitable environment grows at a rate which is proportional to its current population. If a slurry of this yeast has 10 billion cells at time $t = 1$ and 100 billion cells at time $t = 5$, calculate the number of cells in the initial population.

Taking P to be in billions of cells, we have $P(t) = ce^{kt}$, $P(1) = 10 = ce^k$, $P(5) = 100 = ce^{5k}$. Then divide the latter two to obtain $10 = e^{4k}$, so $k = \frac{1}{4}\ln(10)$. Then substitute in $P(1) = 10 = ce^k = ce^{\ln(10)/4}$, and solve to find $c = \frac{10}{e^{\ln(10)/4}}$. Then the initial population is $P(0) = c = \frac{10}{e^{\ln(10)/4}}$.

(If you wish, we may simplify this via log properties to $c = 10^{3/4}$.)

10*. Suppose a rectangular piece of metal is sitting in the sun. It is presently 4 ft wide by 6 ft long. In the heat, it expands in such a way that the perimeter grows at a constant rate of 8 ft/s. If the length is currently increasing by 3 ft/s, find the rate of change of the metal's area.

We have $A = lw$ and $P = 2l + 2w$, so $A' = lw' + wl'$ and $P' = 2l' + 2w'$. We are given $P' = 8\text{ft/s}$, $l' = 3\text{ft/s}$, $w = 4\text{ft}$, and $l = 6\text{ft}$ at the particular moment. Plugging and chugging, we find $w' = 1\text{ft/s}$ and $A' = 18\text{ft}^2/\text{s}$.

11. Consider the function $f(x) = \ln(x)$.
a) Check whether this function satisfies the hypotheses of the mean value theorem on the interval $[1, e]$.
The logarithm is continuous and differentiable on its domain, including the interval $[1, e]$.

b) If it does/did apply, what can/could we conclude?
Hence there is a number c, $1 < c < e$, such that

$$f'(c) = \frac{\ln(e) - \ln(1)}{e - 1} = \frac{1}{e-1}.$$

c) Find a number c as in the theorem, if one exists.

$$f'(c) = \frac{1}{c} = \frac{1}{e-1} \Rightarrow c = e - 1$$

(Note that $1 < e - 1 < e$.)

12. Consider $f(t) = e^{t^2}$. Find the derivative with respect to x of:
a) $\displaystyle\int_0^x f(t)dt$. The Fundamental Theorem directly applies, and we obtain e^{x^2}.
b) $\displaystyle\int_x^8 f(t)dt$.
We first write the integral as $-\displaystyle\int_8^x f(t)dt$, so we have $-e^{x^2}$.
c) $\displaystyle\int_{-3}^{\tan(x)} f(t)dt$.

Let $u = \tan x$, so $\dfrac{d}{dx}\displaystyle\int_{-3}^{\tan(x)} f(t)dt = \dfrac{d}{du}\left(\displaystyle\int_{-3}^{u} f(t)dt\right)\cdot\dfrac{du}{dx} = e^{u^2}\sec^2(x) = e^{\tan^2(x)}sec^2(x)$.

d*) $\displaystyle\int_{x}^{x^3} f(t)dt$.

First split the integral into $\displaystyle\int_{x}^{0} f(t)dt + \int_{0}^{x^3} f(t)dt = -\int_{0}^{x} f(t)dt + \int_{0}^{x^3} f(t)dt$. Then proceeding as in (b) and (c), we obtain $e^{x^6}3x^2 - e^{x^2}$.

13. An armadillo speeds along a straightline path, but is too quick for your speedometer to keep track of continuously. You manage to record the following data:

t	0	1	3	4	7	8
v(t)	2	-1	8	5	-4	-3

a) Use any acceptable Riemann sum with 5 intervals to estimate the position of the armadillo at time $t = 8$ (relative to his starting position).

For example, left endpoints gives $1(2) + 2(-1) + 1(8) + 3(5) + 1(-4) = 19$.

b) Use any acceptable Riemann sum with 5 intervals to estimate the total distance traversed by the armadillo in the interval $[0, 8]$.

Left endpoints here gives $1|2| + 2|-1| + 1|8| + 3|5| + 1|-4| = 31$

14. Find the area of the region bounded by the the curves $y = 2x^3 - 5x^2 + 4x$ and $y = x^3 - 6x^2 + 10x$.
These curves intersect when $2x^3 - 5x^2 + 4x = x^3 - 6x^2 + 10x \Rightarrow x^3 + x^2 - 6x = 0 \Rightarrow x(x+3)(x-2) = 0$.
Testing at $x = -1$ and $x = 1$ tells which curve is above the other in each interval, so we compute

$$\begin{aligned}
&\int_{-3}^{0}(-x^3 - x^2 + 6x)dx + \int_{0}^{2}(x^3 + x^2 - 6x)dx \\
&= \left[-\frac{1}{4}x^4 - \frac{1}{3}x^3 + 3x^2\right]_{-3}^{0} + \left[\frac{1}{4}x^4 + \frac{1}{3}x^3 - 3x^2\right]_{0}^{2} \\
&= \frac{3^4}{4} - 3^2 - 27 + 2^2 + \frac{2^3}{3} - 12
\end{aligned}$$

15. The region bounded by $y = x^3$, $y = 8$, and $x = -1$ is rotated around the line $y = -1$. Find its volume.

Washers:

$$\begin{aligned}
&\pi\int_{x=-1}^{2}\left((8-(-1))^2 - (x^3-(-1))^2\right)dx \\
&= \pi\int -1^2\left(80 - 2x^3 - x^6\right)dx \\
&= \pi\left[80x - \frac{1}{2}x^4 - \frac{1}{7}x^7\right]_{-1}^{2} \\
&= \pi\left(160 - 2^3 - \frac{2^7}{7} - 80 - \frac{1}{2} + \frac{1}{7}\right)
\end{aligned}$$

Shells:

$$\begin{aligned}
&2\pi\int_{y=-1}^{8}(y-(-1))\left(y^{1/3}-(-1)\right)dy \\
&= 2\pi\int_{-1}^{8}\left(y^{4/3} - y - y^{1/3} + 1\right)dy \\
&= 2\pi\left[\frac{3}{7}y^{7/3} - \frac{1}{2}y^2 - \frac{3}{4}y^{4/3} + y\right]_{-1}^{8} \\
&= 2\pi\left(\frac{3\cdot 2^7}{7} - 32 - 12 + 8 + \frac{3}{7} - \frac{1}{2} - \frac{3}{4} + 1\right)
\end{aligned}$$

16. Our squirrel friend pushes his ice cream treat along his fence. Due to the (very odd) rough nature of the fence's top surface, he needs to exert a force of $\frac{1}{(1+x)^2}$ pounds to move the ice cream when he is x meters from his starting position. How much work does he do to move the ice cream to his hideout 5 meters from where he began?

$$W = \int_{x=0}^{5}\frac{dx}{(1+x)^2} = \int_{u=1}^{6}\frac{du}{u^2} = \left[-\frac{1}{u}\right]_{1}^{6} = \frac{5}{6}.$$

17. Consider the function $f(x) = 3x^2$. Find all c, $-2 < c < 4$, such that $f(c)$ equals the average value of $f(x)$ on the interval $[-2, 4]$. (Are we guaranteed to find any?)

Since f is continuous, we are guaranteed to find at least one such c. The average value is

$$\frac{1}{4-(-2)} \int_{-2}^{4} 3x^2 dx = \frac{1}{6} \left[x^3\right]_{-2}^{4} = \frac{72}{6} = 12.$$

So we need $f(c) = 3c^2 = 12$, so that $c = \pm 2$. Only $c = +2$ is in the desired interval (although the distinction between including or excluding $c = 2$ is a subtle one).

18*. Find an expression for the arc length of the curve $y = \frac{\ln(\sin(x))}{\tan(x)}$ between $x = \frac{\pi}{4}$ and $x = \frac{\pi}{2}$. (Don't try to evaluate the mess!)

$$\int_{\pi/4}^{\pi/2} \sqrt{1 + \left(\frac{\tan(x)\frac{\cos(x)}{\sin(x)} - \ln(\sin(x))\sec^2(x)}{\tan^2(x)} \right)^2} \, dx.$$

This is fine, or you could also notice that $\tan(x)\dfrac{\cos(x)}{\sin(x)} = 1$ for a slight simplification.

19*. Consider the curve $y = \sqrt{1-x^2}$.
a) Set up an integral which represents the arc length of this curve between $x = \frac{1}{2}$ and $\frac{\sqrt{2}}{2}$.

$$\int_{1/2}^{\sqrt{2}/2} \sqrt{1 + \left(\frac{-2x}{2\sqrt{1-x^2}}\right)^2}\, dx$$

b) You can directly evaluate this integral. Do so.

$$= \int_{1/2}^{\sqrt{2}/2} \sqrt{1 + \frac{x^2}{1-x^2}} dx = \int_{1/2}^{\sqrt{2}/2} \sqrt{\frac{1}{1-x^2}} dx = \left[\sin^{-1}(x)\right]_{1/2}^{\sqrt{2}/2} = \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}$$

c) Use a geometric argument to verify your answer to part (b).
We were asked for one twelfth of the length of the top half of a unit circle.

d) Consider the solid obtained by rotating the region bounded by this curve and the x-axis around the x-axis. Set up an integral which represents this volume. (Try both methods.)

Washers (Disks):

$$\pi \int_{x=-1}^{1} \left(\sqrt{1-x^2}\right)^2 dx$$

Shells:

$$2\pi \int_{y=0}^{1} (y)\left(2\sqrt{1-y^2}\right) dy$$

e) You can evaluate both of these integrals by hand. Do you get the same answer?

Washers (Disks):

$$= \pi \int_{-1}^{1} (1-x^2) dx$$
$$= \pi \left[x - \frac{1}{3}x^3\right]_{-1}^{1}$$
$$= \pi \left(1 - \frac{1}{3} + 1 - \frac{1}{3}\right)$$
$$= \frac{4\pi}{3}$$

Shells:

take $u = 1 - y^2$, so $du = -2ydy$

$$= -2\pi \int_{u=1}^{0} \sqrt{u} du$$
$$= -2\pi \frac{2}{3} \left[u^{3/2}\right]_1^0$$
$$= \frac{4\pi}{3}.$$

f) You can also interpret the volume of the solid geometrically. Do so to verify your answer to part (e).
This is the volume of the unit sphere.