MATH 221 / Fall 2009

Disclaimer: This review was written by your TA, Ben Reiniger. The final exam will be written by Dr. Reznick. As such, these problems should not be taken as a complete representative of the actual final; rather, they are meant to be a supplement to your review of previous homework, quiz, and exam experiences. Problems marked with an asterisk are somehow different than what you've seen before: more challenging, more interesting, or otherwise.

Name:

1. Evaluate:

i) $\lim_{x\to 4} \frac{(x-4)^2}{(x-4)^4}$

ii) $\lim_{x\to 4} \frac{(x-4)^{253}}{(x-4)^{200}}$

iii) $\lim_{x\to 4} \frac{8(x-4)^{3.14}}{(x-4)^{3.14}}$

iv) $\lim_{x\to 4^+} \frac{\ln{(x-4)}}{x-4}$

v*) $\lim_{x\to 4^+} \frac{\ln(x-4)}{\tan\left(\frac{\pi x}{8}\right)}$

vi) $\lim_{x\to 4} \frac{(x-5)^4}{(x-3)^{12}}$

vii) $\lim_{x\to 0} x^2 \cos\frac{1}{x}$
(Hint: Squeeze!)

2. a) What is the formal definition of the statement "$\lim_{x\to 3} 3x-4=5$"?

b) Prove the statement.

3. A squirrel runs along a fence with position at time t given by $x(t) = 2t^{3/2}$.

a) What is his average velocity between times $t = 1$ and $t = 4$?

b) What is his instantaneous velocity at time $t = 1$?

c) What is the equation for the tangent line to the curve $x(t)$ at $t = 1$?

d) Use a linear approximation based at $t = 1$ to estimate the squirrel's position at time $t = 2$.

4. The squirrel now leaps from his perch toward a passerby with ice cream. In an interesting happenstance of nature, the squirrel's height above his target (factoring in air resistance) is exactly $y = 16 - 2x^{3/2}$, where x is the horizontal distance traveled. What is the distance traveled by the flying rodent?

5. a) Find a function $f(x)$ and a number a so that the following limit represents the definition of $f'(a)$:

$$\lim_{h \to 0} \frac{\sqrt{8+h} - \sqrt{8}}{h}$$

b) Use your answer to (a), along with your knowledge of the derivatives of elementary functions, to evaluate the limit.

c) Now evaluate the limit directly (hint: rationalize the numerator). Do NOT use L'Hopital's Rule.

6. Consider the function $f(x) = x - \ln(x)$.
a) What is the domain of f?
b) Find intervals of increase, decrease, concave up, and concave down.

c) Find and classify local extrema.

d) Find inflection points.

e) Do we have a theorem which guarantees absolute extrema on $(0, 2]$? Find them if they exist.

f) Do we have a theorem which guarantees absolute extrema on $\left[\frac{1}{e^3}, 3\right]$? Find them if they exist.

7. Where is the function

$$f(x) = \begin{cases} \sin x & \text{for } x < -\pi \\ -1 & \text{for } x = -\pi \\ x + \pi & \text{for } -\pi < x \leq 1 \\ -4x + \pi & \text{for } 1 < x \end{cases}$$

continuous? Justify your answers.

8. Find the derivative of $y = x^{\tan^{-1} x}$.

9. Suppose the number of yeast cells in a suitable environment grows at a rate which is proportional to its current population. If a slurry of this yeast has 10 billion cells at time $t = 1$ and 100 billion cells at time $t = 5$, calculate the number of cells in the initial population.

10*. Suppose a rectangular piece of metal is sitting in the sun. It is presently 4 ft wide by 6 ft long. In the heat, it expands in such a way that the perimeter grows at a constant rate of 8 ft/s. If the length is currently increasing by 3 ft/s, find the rate of change of the metal's area.

11. Consider the function $f(x) = \ln(x)$.

a) Check whether this function satisfies the hypotheses of the mean value theorem on the interval $[1, e]$.

b) If it does/did apply, what can/could we conclude?

c) Find a number c as in the theorem, if one exists.

12. Consider $f(t) = e^{t^2}$. Find the derivative with respect to x of:

a) $\displaystyle\int_0^x f(t)dt$.

b) $\displaystyle\int_x^8 f(t)dt$.

c) $\displaystyle\int_{-3}^{\tan(x)} f(t)dt$.

d*) $\displaystyle\int_x^{x^3} f(t)dt$.

$\sqrt{169}$**.** An armadillo speeds along a straightline path, but is too quick for your speedometer to keep track of continuously. You manage to record the following data:

t	0	1	3	4	7	8
v(t)	2	-1	8	5	-4	-3

a) Use any acceptable Riemann sum with 5 intervals to estimate the position of the armadillo at time $t = 8$ (relative to his starting position).

b) Use any acceptable Riemann sum with 5 intervals to estimate the total distance traversed by the armadillo in the interval $[0, 8]$.

14. Find the area of the region bounded by the the curves $y = 2x^3 - 5x^2 + 4x$ and $y = x^3 - 6x^2 + 10x$.

$\lfloor 5\pi \rfloor$**.** The region bounded by $y = x^3$, $y = 8$, and $x = -1$ is rotated around the line $y = -1$. Find its volume.

16. Our squirrel friend pushes his ice cream treat along his fence. Due to the (very odd) rough nature of the fence's top surface, he needs to exert a force of $\frac{1}{(1+x)^2}$ pounds to move the ice cream when he is x meters from his starting position. How much work does he do to move the ice cream to his hideout 5 meters from where he began?

17. Consider the function $f(x) = 3x^2$. Find all c, $-2 < c < 4$, such that $f(c)$ equals the average value of $f(x)$ on the interval $[-2, 4]$. (Are we guaranteed to find any?)

18*. Find an expression for the arc length of the curve $y = \frac{\ln(\sin(x))}{\tan(x)}$ between $x = \frac{\pi}{4}$ and $x = \frac{\pi}{2}$. (Don't try to evaluate the mess!)

19*. Consider the curve $y = \sqrt{1 - x^2}$.

a) Set up an integral which represents the arc length of this curve between $x = \frac{1}{2}$ and $\frac{\sqrt{2}}{2}$.

b) You can directly evaluate this integral. Do so.

c) Use a geometric argument to verify your answer to part (b).

d) Consider the solid obtained by rotating the region bounded by this curve and the x-axis around the x-axis. Set up an integral which represents this volume. (Try both methods.)

e) You can evaluate both of these integrals by hand. Do you get the same answer?

f) You can also interpret the volume of the solid geometrically. Do so to verify your answer to part (e).