Name: Solutions

- **READ THE FOLLOWING DIRECTIONS!**
- **Do NOT open the exam until instructed to do so.**
- You have until 4:30pm to complete this exam. When you are told to stop writing, do it or you will lose all points on the page you write on.
- You may not communicate with other students during this test.
- No written materials of any kind are allowed. No scratch paper is allowed except as given by the proctors.
- No phones, calculators, or any other electronic devices are allowed for any reason, including checking the time (a simple wristwatch is fine).
- Any case of cheating will be taken extremely seriously.
- Show all your work and explain your answers.
- Before turning in your exam, check to make certain you've answered all the questions.
- Make sure to check whether you are asked to *set up* or to *evaluate* integrals.

Some possibly useful formulas:

$$\cos^2 t = \frac{1}{2}(1+\cos(2t))$$
$$\sin^2 t = \frac{1}{2}(1-\cos(2t))$$
$$\sin(2t) = 2\sin(t)\cos(t)$$

1. Consider the planes $2x - y + z = 4$ and $x + 3y - 2z = 9$.

   (a) Are these planes parallel, perpendicular, or neither? How do you know?

   Neither: $\langle 2,-1,1\rangle \neq c\langle 1,3,-2\rangle$ for any $c \Rightarrow$ not parallel

   $\langle 2,-1,1\rangle \cdot \langle 1,3,-2\rangle = 2-3-2 \neq 0 \Rightarrow$ not perpendicular

   (b) If they are parallel, find the distance between them. Otherwise find the equation of the line that is their intersection.

   vector parallel to both $= \vec{v} = \langle 2,-1,1\rangle \times \langle 1,3,-2\rangle$

   $$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -1 & 1 \\ 1 & 3 & -2 \end{vmatrix} = \langle 2-3, -(-4-1), 6+1\rangle = \langle -1, 5, 7\rangle$$

   point in common $= \vec{p}$: $\begin{cases} 2x-y+z=4 \\ x+3y-2z=9 \end{cases}$ $x=0 \Rightarrow \begin{cases} -y+z=4 \\ 3y-2z=9 \end{cases} \Rightarrow -2y+3y = 8+9$, $y=17$, $\Rightarrow z=21$

   $$\ell(t) = \vec{p} + t\,\vec{v} = \langle 0,17,21\rangle + t\langle -1,5,7\rangle.$$

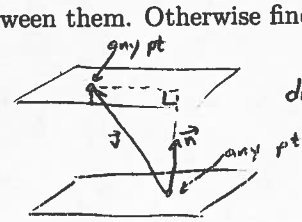
2. Consider the planes $2x - y + z = 4$ and $-4x + 2y - 2z = 7$.

   (a) Are these planes parallel, perpendicular, or neither? How do you know?

   Parallel: $-2\cdot\langle 2,-1,1\rangle = \langle -4,2,-2\rangle$.

   (b) If they are parallel, find the distance between them. Otherwise find the equation of the line that is their intersection.

   One way to do this:



   distance $= \|\mathrm{proj}_{\vec{n}}\vec{v}\|$

   $= \|\vec{v}$ push along $\vec{n}\|$

   $P_1 = \langle 0,0,4\rangle$ $\quad \vec{v} = P_2 - P_1 = \langle 9,0,-4\rangle$

   $P_2 = \langle 9,0,0\rangle$

   $$\|\mathrm{proj}_{\vec{n}}\vec{v}\| = \left\|\frac{\vec{v}\cdot\vec{n}}{\vec{n}\cdot\vec{n}}\,\vec{n}\right\| = \left|\frac{\vec{v}\cdot\vec{n}}{\|\vec{n}\|}\right| \cdot \left\|\frac{\vec{n}}{\|\vec{n}\|}\right\| = \left|\frac{\vec{v}\cdot\vec{n}}{\|\vec{n}\|}\right|$$

   $\vec{n} = \langle 2,-1,1\rangle$ $\quad \vec{v}\cdot\vec{n} = 18 - 0 - 4 = 14$

   $\|\vec{n}\| = \sqrt{4+1+1} = \sqrt{6}$

   $$\text{distance} = \frac{14}{\sqrt{6}}.$$

3. A particle moves in space, with position at time t given by (t, t^2, t^3).

(a) Find the velocity at time $t = 1$.

$\vec{v} = \frac{d}{dt}$ position $= \langle 1, 2t, 3t^2 \rangle$ at $t=1$: $\langle 1,2,3 \rangle$

(b) Find the acceleration at time $t = 1$.

$\vec{a} = \frac{d}{dt} \vec{v} = \langle 0, 2, 6t \rangle$ at $t=1$: $\langle 0,2,6 \rangle$

(c) Find the tangential and normal components of acceleration at time $t = 1$. Is the particle speeding up or slowing down at time $t = 1$?

$$\vec{a}_{tan} = \text{proj}_{\vec{v}} \vec{a} = \vec{a} \text{ pushalong } \vec{v} = \frac{\vec{a} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v}$$

$$= \frac{0+4+18}{1+4+9} \langle 1,2,3 \rangle$$

$$= \frac{11}{7} \langle 1,2,3 \rangle .$$

$$\vec{a}_{normal} = \vec{a} - \vec{a}_{tan} = \langle 0, \tfrac{14}{7}, \tfrac{42}{7} \rangle - \langle \tfrac{11}{7}, \tfrac{22}{7}, \tfrac{33}{7} \rangle$$

$$= \frac{1}{7} \langle -11, -8, 9 \rangle .$$

Since $\vec{a}_{tan} = +\frac{11}{7} \vec{v}$, the particle is speeding up.

4. Let R be the region bounded by the curves $xy = 1$, $xy = 2$, $y = x$, and $y = 3x$. in 1st quadrant Transform $\iint_R x^2\,dx\,dy$ into an integral over a rectangle, and evaluate

$\frac{y}{x}=1$ $\frac{y}{x}=3$

Let $u = xy$, $v = \frac{y}{x}$. So $1 \le u \le 2$, $1 \le v \le 3$.

Method 1: Solve for x, y:

$uv = y^2$ $\frac{u}{v} = x^2$

$y = \sqrt{uv}$ $x = \sqrt{u/v}$ (Note $x, y > 0$, so use $+\sqrt{\ }$)

Area conversion / Jacobian

$$A_{xy}(u,v) = \begin{array}{c|cc|} & u & v \\ x & \frac{1}{2\sqrt{uv}} & -\frac{1}{2}\frac{\sqrt{u}}{v^{3/2}} \\ y & \frac{1}{2}\frac{\sqrt{v}}{\sqrt{u}} & \frac{1}{2}\frac{\sqrt{u}}{\sqrt{v}} \end{array}$$

$$= \frac{1}{4}\left(\frac{1}{v} + \frac{1}{v}\right)$$

$$= \frac{1}{2v}.$$

Method 2:

Area conversion

$$A_{xy}(u,v) = \frac{1}{A_{uv}(x,y)}$$

$$= \left(\begin{array}{c|cc|} & x & y \\ u & y & x \\ v & -y/x^2 & 1/x \end{array} \right)^{-1}$$

$$= \left(\frac{y}{x} + \frac{y}{x}\right)^{-1}$$

$$= \frac{x}{2y}$$

$$= \frac{1}{2v}$$

$$\iint_R x^2\,dx\,dy = \int_1^3 \int_1^2 \underbrace{\left(\frac{u}{v}\right)}_{x^2} \cdot \underbrace{\left|\frac{1}{2v}\right|}_{J}\,du\,dv.$$ ($v > 0$, so $\left|\frac{1}{2v}\right| = \frac{1}{2v}$)

$$= \frac{1}{4}\int_1^3 \frac{(4-1)}{v^2}\,dv$$

$$= \frac{3}{4}\left[-\frac{1}{v}\right]_1^3$$

$$= \frac{1}{2}$$

5. Compute the flow of $\mathbf{F}(x,y,z) = \langle yz^2 + e^z + x, ze^z + x + y, xe^x + xy + z\rangle$ across the sphere S given by $x^2 + y^2 + z^2 = 1$. Which direction is it?

$$\operatorname{div} F = 1 + 1 + 1 = 3$$

Divergence Thm $\Rightarrow$ flow across $= \iiint_{\substack{\text{inside} \\ S}} 3 \, dx\,dy\,dz$

$$= 3 \iiint_{\substack{\text{inside} \\ S}} 1 \, dx\,dy\,dz$$

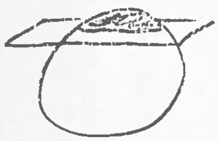
$$= 3 \ \text{Volume}\left(\substack{\text{inside} \\ S}\right)$$

$$= 3 \cdot \frac{4}{3}\pi (1)^3$$

$$= 4\pi.$$

Flow is outward.

6. Compute the volume of the "polar cap" bounded above by the hemisphere $z = \sqrt{25 - x^2 - y^2}$ and below by the plane $z = 4$. (Hint: this is not a "rectangle" in any of our coordinate systems: some variables have to depend on others.)



intersection: $4 = \sqrt{25 - x^2 - y^2}$

$x^2 + y^2 = 9, \ z = 4$

Cylindrical

$4 \le z \le \sqrt{25 - r^2}$

$0 \le r \le 3$

$0 \le \theta \le 2\pi$

$$\text{Volume} = \iiint 1 \, dx\,dy\,dz$$

$$= \int_0^{2\pi}\int_0^3\int_4^{\sqrt{25-r^2}} r \, dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3 \left(r\sqrt{25-r^2} - 4r\right) dr\,d\theta \qquad u = 25 - r^2, \ du = -2r\,dr$$

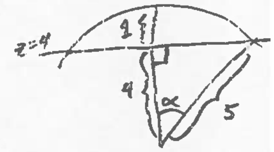
$$= \int_0^{2\pi} \left(\int_{25}^{16} -\tfrac{1}{2}\sqrt{u}\,du - 2(3)^2 \right) d\theta$$

$$= \int_0^{2\pi} \left(\left[\tfrac{1}{3} u^{3/2}\right]_{16}^{25} - 18 \right) d\theta$$

$$= \boxed{2\pi \left(\frac{125 - 64}{3} - 18 \right)}$$

$$= \frac{2\pi}{3}(7)$$

Spherical



$z = \rho\cos\varphi = 4 \iff \rho = 4\sec\varphi$

$4\sec\varphi \le \rho \le 5$

$0 \le \varphi \le \alpha = \cos^{-1}\left(\frac{4}{5}\right)$

$0 \le \theta \le 2\pi$

$$\text{Volume} = \iiint 1 \, dx\,dy\,dz$$

$$= \int_0^{2\pi}\int_0^{\alpha}\int_{4\sec\varphi}^{5} \rho^2 \sin\varphi \, d\rho\,d\varphi\,d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}\int_0^{\alpha} \left(125\sin\varphi - 64\frac{\sin\varphi}{\cos^3\varphi}\right) d\varphi\,d\theta \qquad u = \cos\varphi, \ du = -\sin\varphi\,d\varphi$$

$$= \frac{1}{3}\int_0^{2\pi} \left(\left[-125\cos\varphi\right]_0^{\alpha} + 64\int_1^{4/5} \frac{1}{u^3}\,du \right) d\theta \qquad \left[-\tfrac{1}{2}u^{-2}\right]_1^{4/5}$$

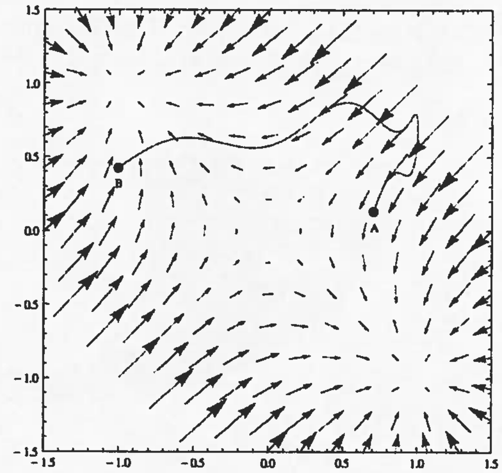
$$= \boxed{\frac{2\pi}{3}\left(-125\cdot\frac{4}{5} + 125 + 32\left(1 - \frac{1}{(4/5)^2}\right)\right)}$$

$$= \frac{2\pi}{3}\left(25 - 32\cdot\frac{9}{16}\right)$$

$$= \frac{2\pi}{3}(7)$$

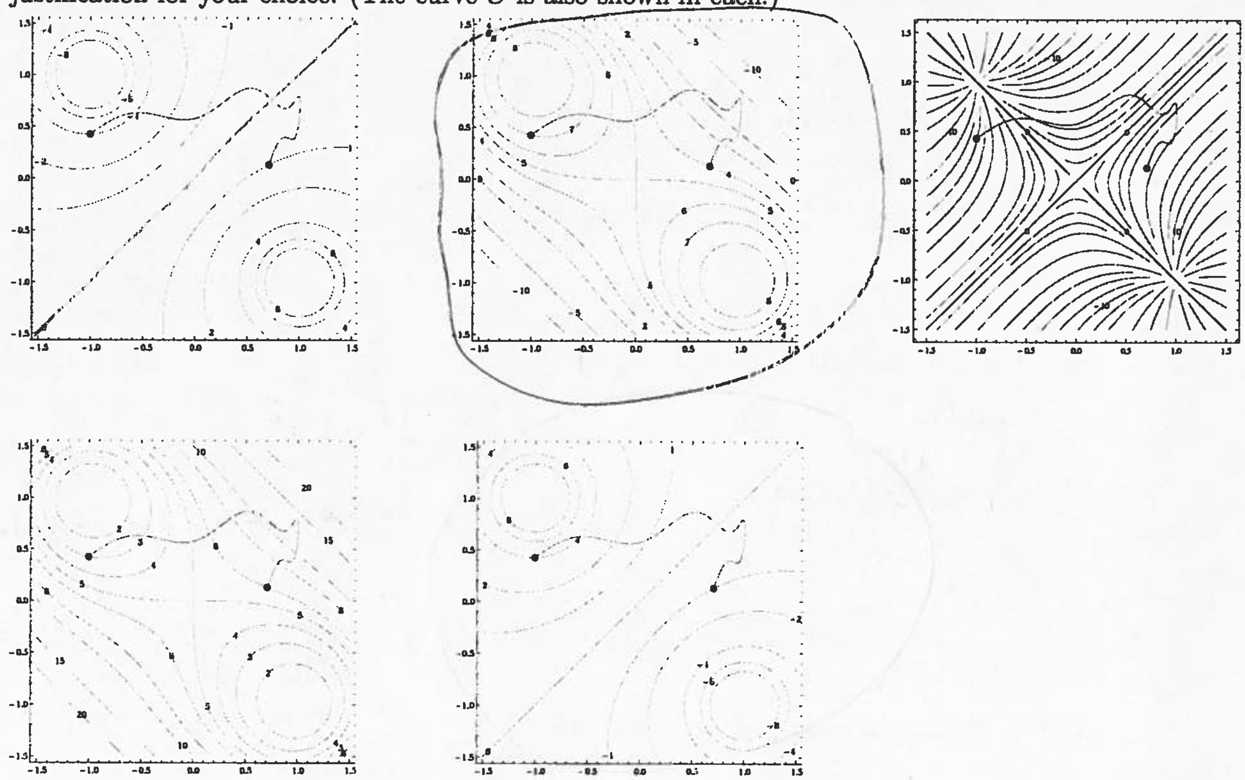
Yay!

7. To the right is shown part of a gradient field $\mathbf{F}$, together with a curve C.



gradient points toward maxima

(a) One of the following is the contour plot of a potential function for $\mathbf{F}$. Circle it. Give a brief justification for your choice. (The curve C is also shown in each.)



(b) Find and classify all critical points of the potential function in the region shown.

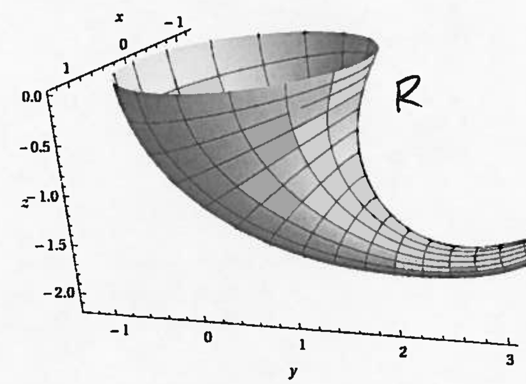
$(-1,1)$ & $(1,-1)$ are maxima; $(0,0)$ is a saddle point

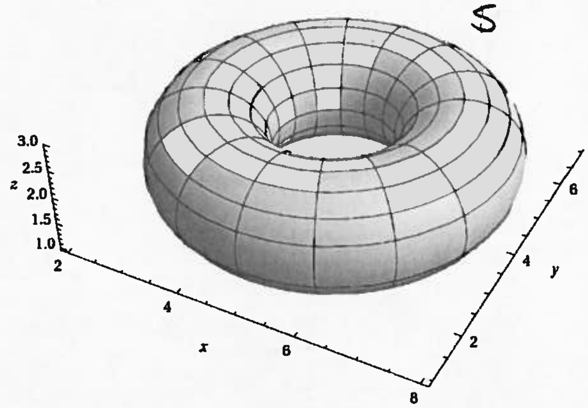
(c) Find $\int_C \mathbf{F} \cdot \langle dx, dy \rangle$, assuming C is parametrized to go from A to B.

$= f(B) - f(A) = 7 - 4 = 3$

where f is the potential function from (a)

8. Below is shown a surface R whose boundary is the unit circle in the xy-plane, and a surface ~~H~~ S that has no boundary. Let $\mathbf{G}(x, y, z) = \langle x, y, z \rangle$. For each of the following, explain your answer.





(a) Which direction is the net flow of $\mathbf{G}$ across S?

$\text{div}\, G = 3$

flow across $= \iiint \text{div} G \quad > 0$

flow is <u>outward</u>.

(b) Which direction is the net flow of $\mathbf{G}$ across R?

$D =$ unit disk in xy-plane.

flow of G across $D = \iint_D G \cdot \vec{dS}$

$\vec{dS}$ on D is $\langle 0, 0, c \rangle$

$z = 0$ on D

$= \iint \langle ?, ?, 0 \rangle \cdot \langle 0, 0, ? \rangle \, du\,dv$

$= 0.$

So flow across R = flow across $R \cup D$ $= \iiint_{\text{inside } R \cup D} \text{div} G > 0$, <u>"outward"</u>.

(c) Which direction is the net flow of curl(G) across R?

Stokes $\Rightarrow$ $\int_{\text{bdry circle}} G \cdot \langle dx, dy, dz \rangle$

circle: $x = \cos t$, $y = \sin t$, $z = 0$, $0 \le t \le 2\pi$

$= \int_0^{2\pi} \langle \cos t, \sin t, 0 \rangle \cdot \langle -\sin t, \cos t, 0 \rangle \, dt$

$= 0.$

(Or: $\text{curl}\, G = \langle 0, 0, 0 \rangle$, so $\iint_R \text{curl}(G) \cdot \vec{dS} = 0$.)