Name: Solutions

- **READ THE FOLLOWING DIRECTIONS!**
- **Do NOT open the exam until instructed to do so.**
- You have until 10:50am to complete this exam. When you are told to stop writing, do it or you will lose all points on the page you write on.
- You may not communicate with other students during this test.
- No written materials of any kind are allowed. No scratch paper is allowed except as given by the proctors.
- No phones, calculators, or any other electronic devices are allowed for any reason, including checking the time (a simple wristwatch is fine).
- Any case of cheating will be taken extremely seriously.
- Show all your work and explain your answers.
- Before turning in your exam, check to make certain you've answered all the questions.
- Make sure to check whether you are asked to *set up* or to *evaluate* integrals.

Some possibly useful formulas:

$$\cos^2 t = \frac{1}{2}(1+\cos(2t))$$
$$\sin^2 t = \frac{1}{2}(1-\cos(2t))$$
$$\sin(2t) = 2\sin(t)\cos(t)$$

Question:	1	2	3	4	Total
Points:	25	25	35	15	100
Score:					

1. (25 points) Compute the flow of $\mathbf{F}(x, y, z) = \langle yz^2 + e^z + x, ze^z + x + y, xe^x + xy + z \rangle$ across the surface R that is the boundary of the solid cube D given by $-1 \le x \le 1$, $-1 \le y \le 1$, $-1 \le z \le 1$. Which direction is it?

$$\text{flow across} = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \text{div}\, F \, dx\, dy\, dz \qquad (\text{Divergence Thm})$$

$$= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} (1+1+1)\, dx\, dy\, dz$$

$$= 3 \cdot \text{Volume}(D)$$

$$= 3 \cdot 2^3$$

2. (25 points) Find the volume contained between the hemispheres $z = \sqrt{4 - x^2 - y^2}$ and $z = \sqrt{1 - x^2 - y^2}$ and the cone $z = \sqrt{x^2 + y^2}$.

Region in spherical coord's is $1 \le \rho \le 2$

$0 \le \varphi \le \frac{\pi}{4}$

$0 \le \theta \le 2\pi$

$$\text{Volume} = \int_0^{2\pi}\int_0^{\pi/4}\int_1^2 (\rho^2 \sin\varphi)\, d\rho\, d\varphi\, d\theta$$

$$= \frac{1}{3}(2^3 - 1^3) \cdot \left(1 - \frac{\sqrt{2}}{2}\right) \cdot 2\pi$$

$$= \frac{14}{3}\pi\left(1 - \frac{\sqrt{2}}{2}\right)$$

3. (35 points) Consider the surface R that is the part of the cone $z = \sqrt{x^2+y^2}$ with $z \le 1$. Let $\mathbf{F}(x,y,z) = \langle yz, -xz, 1\rangle$.

(a) Directly compute $\iint_R \operatorname{curl}(\mathbf{F}) \cdot \mathbf{dS}$. Use a downward/outward normal vector.

Parametrize surface: $x = r\cos\theta$, $y = r\sin\theta$, $z = \sqrt{x^2+y^2} = r$, $\quad 0 \le r \le 1$, $0 \le \theta \le 2\pi$

$$\vec{dS} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \langle -r\cos\theta, -r\sin\theta, r\rangle$$

upward; switch sign

$$\operatorname{curl}(F) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ yz & -xz & 1 \end{vmatrix} = \langle x, y, -2z\rangle$$

$$\iint_R \operatorname{curl}(F) \cdot (-\vec{dS}) = \int_0^{2\pi}\int_0^1 \langle r\cos\theta, r\sin\theta, -2r\rangle \cdot \langle r\cos\theta, r\sin\theta, -r\rangle \, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 3r^2 \, dr\, d\theta = 2\pi.$$

(b) Check your answer to (a) using Stokes's Theorem.

Boundary of R is circle $x^2+y^2=1$, $z=1$.

For the parametrizations to "match," need to go clockwise when viewed from above.

$x = \sin t$, $y = \cos t$, $z = 1$, $\quad 0 \le t \le 2\pi$

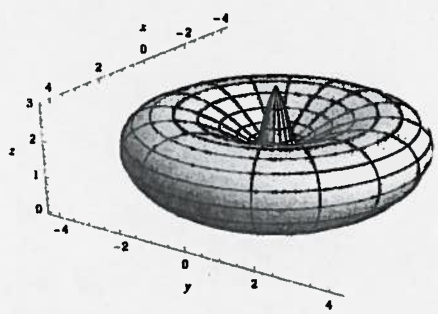
$$\int_C F \cdot \langle dx, dy, dz\rangle,$$

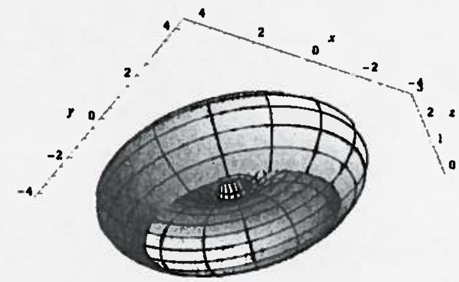
$$= \int_0^{2\pi} \langle \cos t \cdot 1, -\sin t \cdot 1, 1\rangle \cdot \langle \cos t, -\sin t, 0\rangle \, dt$$

$$= \int_0^{2\pi} 1 \, dt$$

$$= 2\pi \quad \checkmark$$

4. (15 points) Let $\mathbf{F}(x,y,z) = \langle x, -y+z, y^2\rangle$. Set up an integral (that you could feed into Mathematica) to compute the flow of $\mathbf{F}$ across the surface R shown below. R can be described in spherical coordinates by $\rho = 1.5(2-\sin(4\varphi))$ (θ free), $0 \le \varphi \le \pi/2$. (In Mathematica's spherical notation, that's $r[s\_, t\_] = 1.5(2 - Sin[4s])$, $0 \le s \le \pi/2$.) Carefully explain your work. (Hint: don't work too hard.)





$\operatorname{div} F = 1 - 1 + 0 = 0,$

so seek a substitute surface.

Boundary of R is where $\varphi = \frac{\pi}{2}$, so $\rho = 1.5(2-0) = 3.$

$\hookrightarrow$ xy-plane

So the disk D of radius 3 in the xy-plane will work.

Parametrize D: $x = r\cos\theta$, $y = r\sin\theta$, $z = 0$, $0 \le r \le 3$, $0 \le \theta \le 2\pi$

$$\vec{dS} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \langle 0, 0, r\rangle.$$

$$\iint_D F \cdot \vec{dS} = \int_0^{2\pi}\int_0^3 \langle r\cos\theta, -r\sin\theta + 0, r^2\sin^2\theta\rangle \cdot \langle 0,0,r\rangle \, dr\, d\theta$$

$$= \boxed{\int_0^{2\pi}\int_0^3 r^3 \sin^2\theta \, dr\, d\theta}$$

Scratch Paper - Do Not Remove