Name: Solutions

- **READ THE FOLLOWING DIRECTIONS!**
- **Do NOT open the exam until instructed to do so.**
- You have until 10:50am to complete this exam. When you are told to stop writing, do it or you will lose all points on the page you write on.
- You may not communicate with other students during this test.
- No written materials of any kind are allowed. No scratch paper is allowed except as given by the proctors.
- No phones, calculators, or any other electronic devices are allowed for any reason, including checking the time (a simple wristwatch is fine).
- Any case of cheating will be taken extremely seriously.
- Show all your work and explain your answers.
- Before turning in your exam, check to make certain you've answered all the questions.

Some possibly useful formulas:

$$\cos^2 t = \frac{1}{2}(1 + \cos(2t))$$
$$\sin^2 t = \frac{1}{2}(1 - \cos(2t))$$

1. Let $\mathbf{F}(x,y) = \langle ye^x + x^2 + yz^2, e^x + z + xz^2, y + 2xyz + \cos(z)\rangle$.

(a) Verify that $\mathbf{F}$ is a gradient field.

Check that $\text{curl}\vec{F} = \vec{0}$ ($\vec{F}$ has no singularities):

$$\text{curl}(\vec{F}) = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ m & n & p \end{vmatrix} = \Big(\underbrace{(1+2xz)}_{\partial_y p} - \underbrace{(1+2xz)}_{\partial_z n}\Big)\vec{i} - \Big(\underbrace{(2yz)}_{\partial_x p} - \underbrace{(2yz)}_{\partial_z m}\Big)\vec{j}$$

$$+ \Big(\underbrace{(e^x+z^2)}_{\partial_x n} - \underbrace{(e^x+z^2)}_{\partial_y m}\Big)\vec{k} = \langle 0,0,0\rangle$$

(b) Find a potential function for $\mathbf{F}$.

$f(x,y,z)$ such that $\nabla f = \vec{F}$, i.e. $\partial_x f = m$, $\partial_y f = n$, $\partial_z f = p$.

$\partial_x f = m \Rightarrow f(x,y,z) = ye^x + \frac{1}{3}x^3 + xyz^2 + g(y,z)$

$\partial_y f = n \rightarrow e^x + 0 + xz^2 + \partial_y g(y,z) = e^x + z + xz^2$

$\Rightarrow \partial_y g(y,z) = z \Rightarrow g(y,z) = yz + h(z)$

$\partial_z f = p \rightarrow 0 + 0 + 2xyz + y + h'(z) = y + 2xyz + \cos z$

$\Rightarrow h'(z) = \cos z \Rightarrow h(z) = \sin z \ (+c)$

$f(x,y,z) = ye^x + \frac{1}{3}x^3 + xyz^2 + yz + \sin z$.

(c) Compute the flow of $\mathbf{F}$ along the part of the curve $x = \sin(\pi t)$, $y = e^t$, $z = t^3$ going from $(0,1,0)$ to $(0,e,1)$.

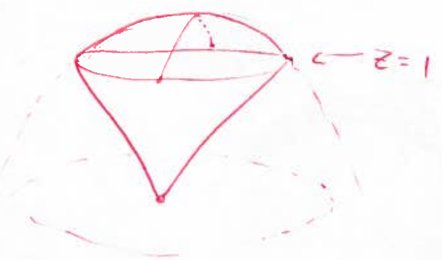
By the Fundamental Thm for Path Integrals,

flow along $= f(0,e,1) - f(0,1,0)$

$= (e + e + \sin(1)) - (1)$

$= 2e + \sin(1) - 1$.

2. Let $\mathbf{F}(x,y,z) = \langle 3xz^2, y^3, 3x^2z \rangle$. The surface R consists of two parts: the part of the cone $z = \sqrt{x^2+y^2}$ with $z \leq 1$ and the part of the sphere $x^2+y^2 = 1$ inside the cone. Compute the flux of $\mathbf{F}$ across R.



Use the Divergence Theorem.

Flux (flow across) R

$$= \iiint_{\text{interior of } R} \operatorname{div} F \, dx\, dy\, dz$$

$$= \iiint_{\text{inside } R} (3z^2 + 3y^2 + 3x^2)\, dx\, dy\, dz$$

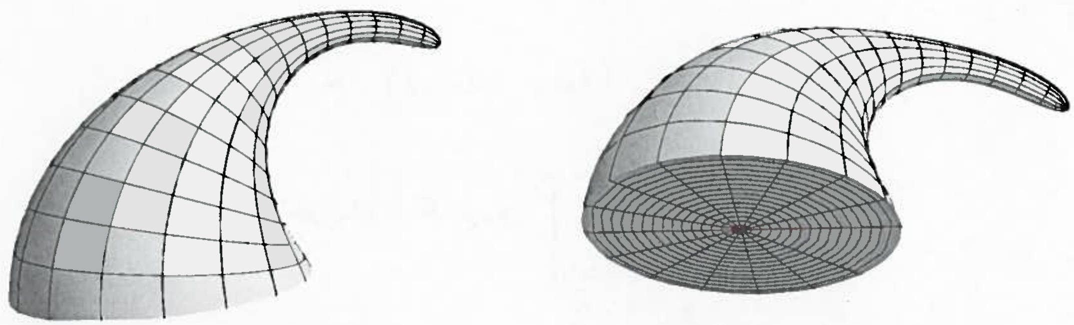
switch to spherical

$$= \int_0^{2\pi}\int_0^{\pi/4}\int_0^{1} 3\rho^2 (\rho^2 \sin\varphi)\, d\rho\, d\varphi\, d\theta$$

$$= \frac{3}{5} \cdot 2\pi \cdot \left[-\cos\varphi\right]_0^{\pi/4}$$

$$= \frac{3}{5} \cdot 2\pi \cdot \left(1 - \frac{\sqrt{2}}{2}\right).$$

3. Let $\mathbf{F}(x,y,z) = \langle xze^{xyz}, -yze^{xyz}, -2\rangle$. The surface R consists of two parts: the disk R_1 given by $x^2+y^2 \le 1$ in the xy-plane, and the "hat" surface R_2. This is pictured below from two different perspectives.



(a) Explain why the flow of $\mathbf{F}$ across R is zero.

$$\operatorname{div} F = (ze^{xyz} + xyz^2e^{xyz}) + (-ze^{xyz} - xyz^2e^{xyz}) + (0) = 0.$$

(b) Compute the flow of $\mathbf{F}$ across R_1. (Which direction is it?)

Note: upward

Parametrise R_1 as $x = r\cos t$, $y = r\sin t$, $z = 0$, $r \in [0,1]$, $t \in [0, 2\pi]$

$$\vec{dS} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos t & \sin t & 0 \\ -r\sin t & r\cos t & 0 \end{vmatrix} = \langle 0,0,r\rangle$$

$$\text{flow across} = \iint_{R_1} F \cdot \vec{dS} = \int_0^{2\pi}\int_0^1 \langle \cdots, \cdots, -2\rangle \cdot \langle 0,0,r\rangle \, dr\, dt$$

$$= \int_0^{2\pi}\int_0^1 -2r \, dr\, dt = -2\pi.$$

So flow is 2π downward.

(c) Find the flow of $\mathbf{F}$ across R_2. (Which direction is it?)

By (a), the 2π flow out of R through R_1 is balanced by 2π flow into R through R_2.

So flow across R_2 is 2π "inward".

4. Let $\mathbf{G} = \langle y, -x, e^{xyz} \rangle$.

(a) Compute curl $\mathbf{G}$.

$$\text{curl } \vec{G} = \nabla \times G = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial_x & \partial_y & \partial_z \\ y & -x & e^{xyz} \end{vmatrix} = \langle xze^{xyz}, -yze^{xyz}, -2 \rangle$$

$= \vec{F}$ from 3) !

(b) Using (a), give another quick solution to question 3(b). (Be careful about whether the sign is correct!)

By Stokes's Theorem,

$$\text{flow of F across } R_1 = \iint_{R_1} F \cdot d\vec{S}$$

$$= \iint_{R_1} \text{curl } G \cdot d\vec{S}$$

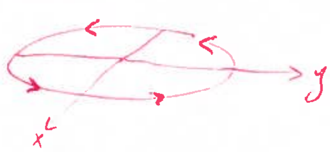
boundary of R_1 is unit disk,

$x = \cos t$

$y = \sin t$ $\quad t \in [0, 2\pi]$

$z = 0$

orientation:



$$= \int_{\text{boundary of } R_1} G \cdot \langle dx, dy, dz \rangle$$

$$= \int_0^{2\pi} \langle \sin t, -\cos t, e^{(\cos t)(\sin t)(0)} \rangle \cdot \langle -\sin t, \cos t, 0 \rangle dt$$

$$= \int_0^{2\pi} (-1)\, dt = -2\pi.$$

by right-hand-rule, normal vector to R_1 $(d\vec{S})$ should point upward,

so net flow is 2π downward

5. A very linear-minded student gets stumped by the following integral. Think nonlinear and compute the integral for them.

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} \sqrt{x^2+y^2+z^2}\,dz\,dy\,dx$$

$$\sqrt{x^2+y^2} \le z \le \sqrt{2-x^2-y^2}$$

inside cone

inside sphere of radius $\sqrt{2}$

$$0 \le y \le \sqrt{1-x^2}$$

"right" side

get whole cone base

$$-1 \le x \le 1$$

whole thing

So region is described in spherical as

$$0 \le \rho \le \sqrt{2}$$

$$0 \le \varphi \le \pi/4$$

$$0 \le \theta \le \pi .$$

Integral becomes

$$\int_0^{\pi}\int_0^{\pi/4}\int_0^{\sqrt{2}} \rho\,(\rho^2 \sin\varphi)\,d\rho\,d\varphi\,d\theta$$

$$= \frac{1}{4}(\sqrt{2})^4 \cdot \pi \cdot \left(1-\frac{\sqrt{2}}{2}\right).$$

6. Find the volume of the region bounded by the planes

$$x+y+z=1 \quad x+2y=-1 \quad y+z=2$$
$$\underbrace{x+y+z}_{u}=4 \quad \underbrace{x+2y}_{v}=1 \quad \underbrace{y+z}_{w}=4$$

$$x = u - w$$

$$y = \frac{1}{2}(-u + v + w)$$

$$z = \frac{1}{2}(u - v + w)$$

$$J = \begin{array}{c c} & \begin{array}{ccc} u & v & w \end{array} \\ \begin{array}{c} x \\ y \\ z \end{array} & \left| \begin{array}{ccc} 1 & 0 & -1 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & +1/2 \end{array} \right| \end{array}$$

$$= \frac{1}{2} + 0 - 0 = 1/2$$

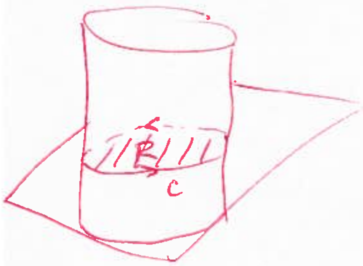
$$\text{Volume} = \iiint_R 1 \, dx\,dy\,dz$$

$$= \int_2^4 \int_{-1}^1 \int_1^4 1/2 \; du\,dv\,dw$$

$$= 2 \cdot 2 \cdot 3 \cdot \frac{1}{2}$$

$$= \boxed{6.}$$

7. Use Stokes's Theorem to compute the flow of $\mathbf{F}(x,y,z) = \langle xy, 2z, 3y\rangle$ along C, where C is the curve of intersection of the cylinder $x^2+y^2=9$ and the plane $x+z=5$. When viewed from above (from large positive z), is the flow clockwise or counterclockwise?



For Stokes, we need a surface whose boundary is C. The part of the plane inside the cylinder will work well.

Parametrize R:

$x = r\cos t$ $\quad 0 \le r \le 3$

$y = r\sin t$ $\quad 0 \le t \le 2\pi$

$z = 5 - r\cos t$

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial x & \partial y & \partial z \\ xy & 2z & 3y \end{vmatrix} = \langle 1, 0, -x\rangle$$

$$\vec{dS} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos t & \sin t & -\cos t \\ -r\sin t & r\cos t & r\sin t \end{vmatrix} = \langle r, 0, r\rangle$$

Note: generally upward, so in Stokes, need boundary parametr counterclockwise when viewed from above. The parametrization above has the correct direction

Stokes: flow along C $= \iint_R \operatorname{curl} F \cdot \vec{dS}$

$$= \int_0^{2\pi}\int_0^3 \langle 1, 0, -r\cos t\rangle \cdot \langle r, 0, r\rangle \, dr\, dt$$

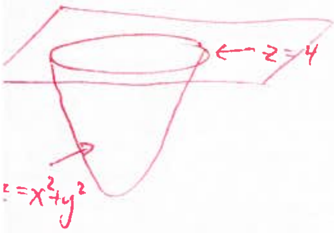
$$= \int_0^{2\pi}\int_0^3 (r - r^2\cos t)\, dr\, dt$$

$$= \int_0^{2\pi} \left(\frac{9}{2} - 9\cos t\right) dt$$

$$= \boxed{9\pi} - 0$$

Counterclockwise when viewed from above.

8. Compute $\iiint_E z\,dV$, where E is enclosed by the surface $z = x^2 + y^2$ and the plane $z = 4$.



Use cylindrical coordinates.

$x^2+y^2 \le z \le 4 \rightsquigarrow r^2 \le z \le 4$

$0 \le r \le 2$

$0 \le \theta \le 2\pi$

$\left[\begin{array}{c} \text{— OR —} \\ 0 \le r \le \sqrt{z} \\ 0 \le z \le 4 \\ 0 \le \theta \le 2\pi \end{array}\right]$

$$\iiint_E z\,dx\,dy\,dz = \int_0^{2\pi}\int_0^{2}\int_{r^2}^{4} z\,(r)\,dz\,dr\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^{2} (16 - r^4)\,r\,dr\,d\theta$$

$$= \boxed{\frac{1}{2}\,(2\pi)\left(32 - \frac{32}{5}\right)}$$