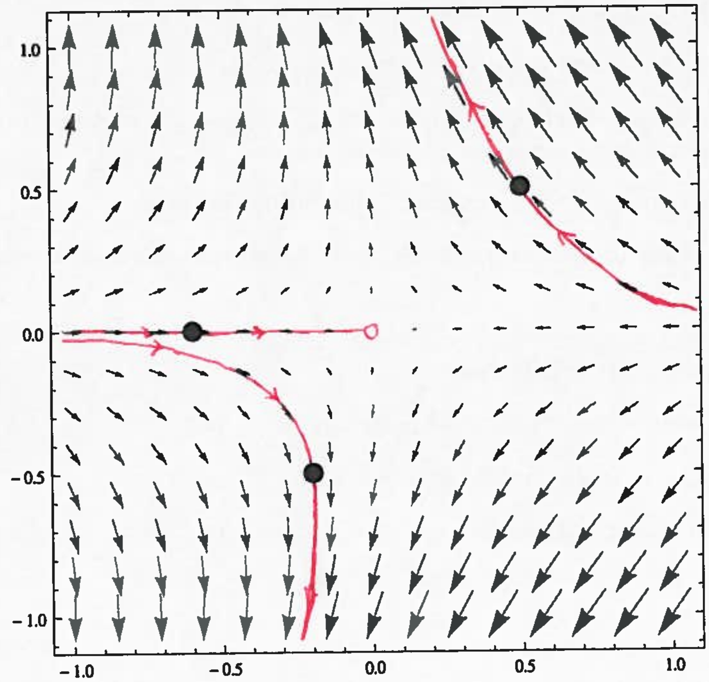
Name: ______________________________

- **READ THE FOLLOWING DIRECTIONS!**
- **Do NOT open the exam until instructed to do so.**
- You have until 10:50am to complete this exam. When you are told to stop writing, do it or you will lose all points on the page you write on.
- You may not communicate with other students during this test.
- No written materials of any kind are allowed. No scratch paper is allowed except as given by the proctors.
- No phones, calculators, or any other electronic devices are allowed for any reason, including checking the time (a simple wristwatch is fine).
- Any case of cheating will be taken extremely seriously.
- Show all your work and explain your answers.
- Before turning in your exam, check to make certain you've answered all the questions.

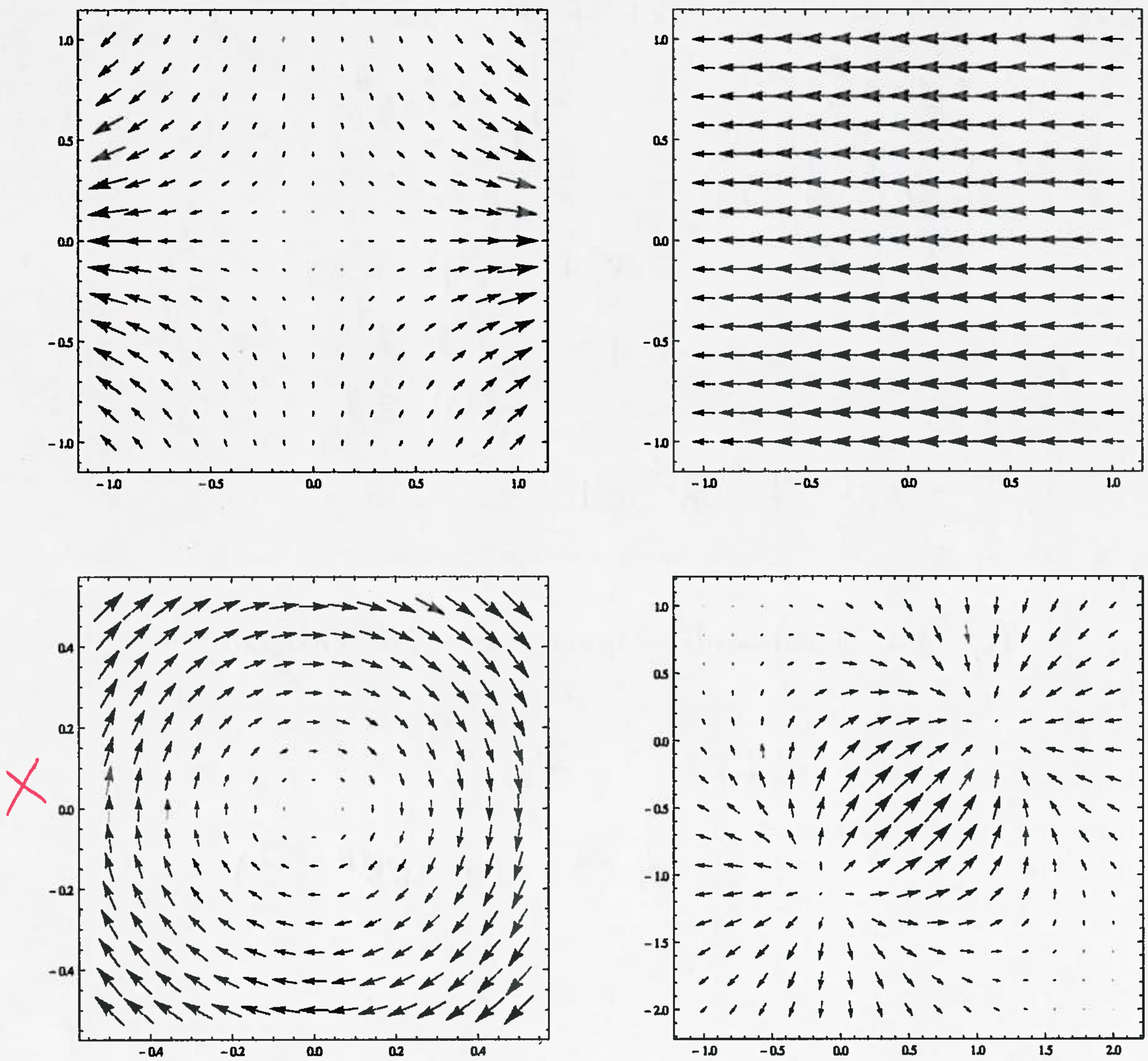
Some possibly useful formulas:

$$\cos^2 t = \frac{1}{2}(1+\cos(2t))$$
$$\sin^2 t = \frac{1}{2}(1-\cos(2t))$$

1. Sketch the trajectories that pass through the indicated points below.



2. All but one of the following vector fields are gradient fields. Which one can't be a gradient field? **Why?**



This one can't be a gradient field:

1) Trajectories are closed curves; or

2) The flow along a circle centered at the origin is clockwise (not zero).

3. Let $\mathbf{F}(x, y) = \langle ye^x + \frac{4}{x^2}, e^x + y^4 \rangle$.

(a) Find a potential function for $\mathbf{F}$. If $F = \nabla f$, then

$$\partial_x f = ye^x + \frac{4}{x^2} \qquad \partial_y f = e^x + y^4$$

$$\Rightarrow f(x,y) = ye^x + \frac{1}{3}x^3 + g(y)$$

$$e^x + 0 + g'(y) = e^x + y^4$$

$$\Rightarrow g'(y) = y^4$$

$$\Rightarrow g(y) = \frac{1}{5}y^5 \ (+C)$$

$$\Rightarrow f(x,y) = ye^x + \frac{1}{3}x^3 + \frac{1}{5}y^5.$$

(b) Compute the flow of $\mathbf{F}$ along the part of the curve $y = \sin(\pi x)$ going from $(0, 0)$ to $(3, -1)$.

By the Fundamental Theorem of Path Integrals,

$$= f(3,-1) - f(0,0)$$

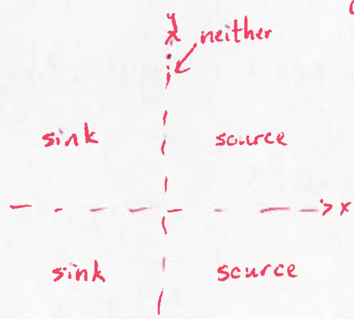
$$= \left(-e^3 - \frac{1}{3} - \frac{1}{5}\right) - (0 - 0 + 0)$$

4. Find all sources and sinks of the vector field $\mathbf{F}(x,y) = \langle y^4, xy^3e^x \rangle$.

$$\operatorname{div}\vec{F} = 0 + 3xy^2e^x;$$

since $3y^2e^x \geq 0$, $\operatorname{div}\vec{F} \overset{\text{source}}{>0}$ when $x>0$ and $\overset{\text{sink}}{<0}$ when $x<0$, & $y \neq 0$

$\operatorname{div}\vec{F} = 0$ when $x=0$ or $y=0$. → neither



5. Find all sources and sinks of the vector field $\mathbf{G}(x,y) = \left\langle \frac{x+y}{x^2+y^2}, \frac{y-x}{x^2+y^2} \right\rangle$.

$$\operatorname{div}\vec{G} = \frac{(x^2+y^2)(1)-(x+y)(2x)}{(x^2+y^2)^2} + \frac{(x^2+y^2)(1)-(y-x)(2y)}{(x^2+y^2)^2} = 0 \text{ except at } (0,0)$$

$$\int_{\text{unit circle}} \vec{G} \cdot \langle dy, -dx \rangle \qquad x=\cos t,\ y = \sin t \quad t \in [0,2\pi]$$

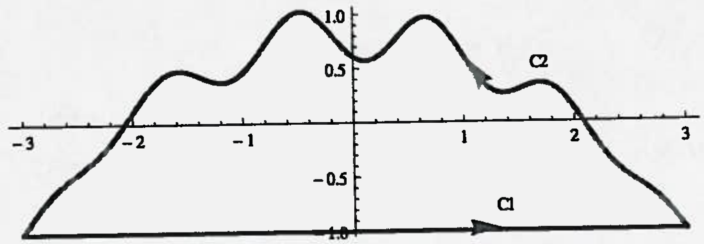
$$= \int_0^{2\pi} \langle \cos t + \sin t, \sin t - \cos t \rangle \cdot \langle \cos t, \sin t \rangle \, dt$$

$$= \int_0^{2\pi} (\cos^2 t + \sin^2 t)\, dt$$

$$= 2\pi$$

So $(0,0)$ is a (the only) source.

6. Let $\mathbf{F}(x, y) = \langle xy^2 \,{}^{+1}, x^2y\rangle$. The curve C consists of two parts, C_1 and C_2, as shown below.



(a) Find the flow of $\mathbf{F}$ along C.

$\text{rot}\, \vec{F} = 2xy - 2xy = 0$, $\vec{F}$ has no singularities,

so flow along is zero.

(b) Find the flow of $\mathbf{F}$ along C_1.

C_1: $x = t$, $y = -1$, $t \in [-3, 3]$

$$\int_{C_1} \vec{F} \cdot \langle dx, dy\rangle = \int_{-3}^{3} \langle t(-1)^2 + 1, t^2(-1)\rangle \cdot \langle 1, 0\rangle \, dt$$

$$= \int_{-3}^{3} t + 1 \, dt$$

$$= 6.$$

(c) Find the flow of $\mathbf{F}$ along C_2.

flow along C = flow along C_1 + flow along C_2,

so flow along C_2 is -6.

7. Let $\mathbf{F}(x,y) = \langle 2, \frac{1}{2}y^2 \rangle$. Let C be the curve going from $(1,0)$ to $(0,2)$ along the parabola $4-4x = y^2$, then from $(0,2)$ to $(-1,0)$ along the parabola $4+4x=y^2$, then from $(-1,0)$ to $(1,0)$ along the x-axis. Let R be the region bounded by C.

(a) Explain why you can measure the flow of $\mathbf{F}$ across C by the double integral

$$\iint_R y\,dx\,dy.$$

Gauss-Green gives

$$\text{flow across} = \int_C F \cdot \langle dy, -dx \rangle = \iint_R \operatorname{div} \vec{F}\, dx\,dy,$$

and $\operatorname{div} \vec{F} = 0 + y$.

(b) Use the transformation $x = u^2 - v^2$ and $y = 2uv$ to compute the above integral.

$$J = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4u^2 + 4v^2$$ (≥ 0, so don't need abs values)

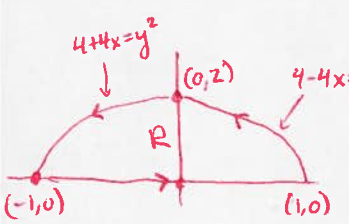
$$\iint_R y\,dx\,dy = \iint_S (2uv)(4u^2+4v^2)\,du\,dv$$

$$= 8\int_0^1 \int_0^1 (u^3 v + uv^3)\,du\,dv$$

$$= 8\int_0^1 \left[\tfrac{1}{4}u^4 v + \tfrac{1}{2}u^2 v\right]_0^1 dv$$

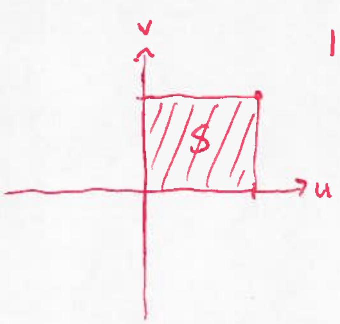
$$= \int_0^1 (2v + 4v)\,dv = 6\left[\tfrac{1}{2}v^2\right]_0^1$$

$$= 3.$$



$$4-4x = y^2 \iff 4 - 4(u^2 - v^2) = 4u^2v^2$$
$$1 - u^2 + v^2 = u^2v^2$$
$$-1 = \frac{1-u^2}{u^2-1} = v^2 \text{ or } u^2 = 1 \qquad u=1$$
(since $u \geq 0$)

$$4+4x = y^2 \iff 4 + 4(u^2 - v^2) = 4u^2v^2$$
$$1 + u^2 - v^2 = u^2v^2$$
$$1 = \frac{1+u^2}{u^2+1} = v^2 \qquad v = 1$$
(since $v \geq 0$)



(x,y)	(u,v)
(0,0)	(0,0)
(1,0)	(1,0)
(-1,0)	(0,1)
(0,2)	(1,1)

8. Let R be the region in the first quadrant bounded by the curves $x^2 - y^2 = 1$, $x^2 - y^2 = 4$, $x - y = 1$, and $x - y = 3$. Compute the area of R.

Take $u = x^2 - y^2 \quad v = x - y \quad (\text{so } 1 \le u \le 4, \ 1 \le v \le 3)$

Then $u = x^2 - y^2 = (x-y)(x+y) = v(x+y)$,

so $x + y = u/v$.

$$\begin{array}{ll} x+y = \frac{u}{v} & x+y = \frac{u}{v} \\ \text{And } +(x-y = v) & -(x-y = v) \\ \hline \Rightarrow 2x = v + \frac{u}{v} & 2y = \frac{u}{v} - v \\ x = \frac{1}{2}\left(v + \frac{u}{v}\right) & y = \frac{1}{2}\left(\frac{u}{v} - v\right). \end{array}$$

So $J = \begin{vmatrix} \frac{1}{2}\cdot\frac{1}{v} & \frac{1}{2}\left(1 - \frac{u}{v^2}\right) \\ \frac{1}{2}\cdot\frac{1}{v} & \frac{1}{2}\left(-\frac{u}{v^2} - 1\right) \end{vmatrix}$

$= \frac{1}{4v}\left[\left(-\frac{u}{v^2} - 1\right) - \left(1 - \frac{u}{v^2}\right)\right]$

$= -\frac{1}{2v}$; $0 < 1 \le v = x - y \le 3$, so $|J| = \frac{1}{2v}$.

$$\text{Area}(R) = \iint_R 1 \, dx\, dy = \iint_S \frac{1}{2v} \, du\, dv$$

$$= \int_1^3 \int_1^4 \frac{1}{2v} \, du\, dv$$

$$= \frac{3}{2} \int_1^3 \frac{1}{v} \, dv$$

$$= \frac{3}{2} \ln|v| \Big|_1^3$$

$$= \frac{3}{2}(\ln 3 - \ln 1)$$

$$= \frac{3}{2} \ln 3.$$

9. Compute $\iint_D \frac{1}{\sqrt{x^2+y^2}}\,dx\,dy$ where D is the unit disk. (Remark: note that this is an improper double integral, but your choice of transformation turns it into a proper (and easily computed) iterated integral.)

Use polar: $\int_0^{2\pi}\int_0^1 \frac{1}{r}\cdot r\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^1 1\,dr\,d\theta$

$= 2\pi$.