Solutions

1. (10 points) Consider the parametric equations $x = t^2$, $y = \sin(\pi t)$.

   (a) Find the slope(s) of the tangent line(s) to the curve at the point(s) where $x = 9$.

   Solution: $x = t^2 = 9$ if and only if $t = \pm 3$. The slope of a parametric curve is given by

   $$\frac{y'(t)}{x'(t)} = \frac{\pi \cos(\pi t)}{2t},$$

   and evaluating this at $t = \pm 3$ we get $\pm\pi/6$.

   (b) What are the minimum and maximum values taken by x and y in this curve?

   Solution: x has minimum 0 and no maximum. y has minimum -1 and maximum $+1$.

2. A certain student is working on a paper. Having a long history of writing papers, she has found that she writes at a rate of $10e^{-t}$ pages per minute, where t is the time since her last break.

(a) (12 points) If she takes a break every hour, how long of a paper can she write in three hours? (That's three hours of writing time; ignore the length of the breaks.)

Solution:

$$3\int_0^{60} 10e^{-t}\,dt = 30\left[-e^{-t}\right]_0^{60} = 30\left(1-e^{-60}\right) \text{ pages.}$$

(b) (12 points) If she takes no break and works for a very very long time, how long of a paper can she write?

Solution:

$$\int_0^{\infty} 10e^{-t}\,dt = \left[-10e^{-t}\right]_0^{\infty} = 10(1-0) = 10 \text{ pages.}$$

3. (15 points) In larger classes, hard exams often have scores that follow a normal distribution. If such an exam has a mean of 200 points and a standard deviation of 30, then the distribution is given by

$$\text{normal}[x, 200, 30] = \frac{1}{30\sqrt{2\pi}} e^{-\left(\frac{x-200}{30\sqrt{2}}\right)^2}.$$

Write an expression in terms of the error function for the percentage of students who score between 210 and 230. As a reminder,

$$\text{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt.$$

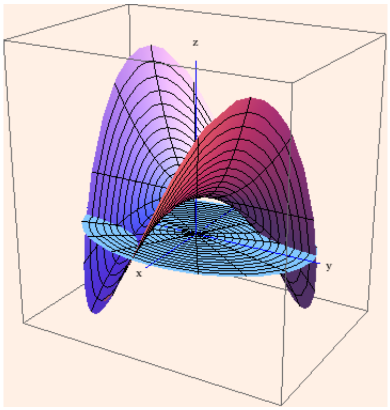
Solution: The proportion of people between 210 and 230 is given by the integral

$$\int_{210}^{230} \frac{1}{30\sqrt{2\pi}} e^{-\left(\frac{x-200}{30\sqrt{2}}\right)^2}\, dx.$$

We want this to look more like the Erf integral, so make a transformation, $u = (x-200)/30\sqrt{2}$. Then $du = 1/30\sqrt{2}\, dx$. The bounds on the integral become $1/3\sqrt{2}$ and $1/\sqrt{2}$, so the integral becomes

$$\begin{aligned}\int_{1/3\sqrt{2}}^{1/\sqrt{2}} \frac{1}{\sqrt{\pi}} e^{-u^2}\, du &= \frac{1}{2}\left(\int_0^{1/\sqrt{2}} \frac{2}{\sqrt{\pi}} e^{-u^2}\, du - \int_0^{1/3\sqrt{2}} \frac{2}{\sqrt{\pi}} e^{-u^2}\, du\right) \\ &= \frac{1}{2}\left(\text{Erf}\left(\frac{1}{\sqrt{2}}\right) - \text{Erf}\left(\frac{1}{3\sqrt{2}}\right)\right).\end{aligned}$$

4. Consider the surface plotted below. It is given by $z = f(x,y)$ over the region R inside the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$, where $f(x,y) = xy + 1$.



(a) (8 points) Without doing any computations, do you think the integral $\iint_R f(x,y)\,dx\,dy$ is positive or negative? Why?

Solution: The integral should be positive, because it appears that there is more space above the region R (and below that surface) than there is below the region R (and above the surface).

(b) (15 points) Compute $\iint_R f(x,y)\,dx\,dy$.

Solution: We can parameterize the boundary of R (counterclockwise, going exactly once around) by $x = 3\cos t$, $y = 2\sin t$, $t \in [0, 2\pi)$. Then, taking $m = 0$ and

$$n = \int_0^x f(s,y)\,ds = \int_0^x (sy+1)\,ds = \left[\frac{1}{2}s^2 y + s\right]_0^x = \frac{1}{2}x^2 y + x,$$

Gauss-Green gives us that

$$\begin{aligned}\iint_R f(x,y)\,dx\,dy &= \int_0^{2\pi} \left(\frac{1}{2}x^2 y + x\right) y'\,dt \\ &= \int_0^{2\pi} \left(9\cos^2(t)\sin(t) + 3\cos(t)\right)\left(2\cos(t)\right)\,dt \\ &= 18\int_0^{2\pi} \cos^3(t)\sin(t)\,dt + 3\int_0^{2\pi} 2\cos^2(t)\,dt\end{aligned}$$

The first integral can be done with a transformation, $u = \cos(t)$. Omitting the details, we get an integral from $u = 1$ to $u = 1$, so the first integral is zero. For the second, we use the trigonometric identity

$$\begin{aligned}&= 0 + 3\int_0^{2\pi} (1 + \cos(2t))\,dt \\ &= \left[3t + \frac{3}{2}\sin(2t)\right]_0^{2\pi} \\ &= 6\pi.\end{aligned}$$

5. (15 points) Solve the differential equation $y' = y \sin x$, $y(0) = e$.

Solution: We separate to get

$$\frac{y'}{y} = \sin x.$$

Using the method from the text, we integrate each side (first changing x to t for easier bookkeeping)

$$\begin{aligned}
\int_0^x \frac{y'(t)}{y(t)}\,dt &= \int_0^x \sin t\,dt \\
\log(y(t))\big|_0^x &= -\cos t\big|_0^x \\
\log(y(x)) - \log(y(0)) &= -\cos(x) + 1 \\
\log(y(x)) - \log(e) &= 1 - \cos(x) \\
\log(y(x)) &= 2 - \cos(x) \\
y(x) &= e^{2-\cos(x)}.
\end{aligned}$$

6. (8 points) Compute $\int_0^t xe^{3x}\,dx$.

Solution: There are two good choices here: integration by parts and integration by differentiation. For integration by parts, take $u = x$, $dv = e^{3x}\,dx$. Then $du = dx$ and $v = \frac{1}{3}e^{3x}$, so the integral becomes

$$\left[\frac{1}{3}xe^{3x}\right]_0^t - \int_0^t \frac{1}{3}e^{3x}\,dx = \left[\frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x}\right]_0^t$$
$$= \frac{1}{3}te^{3t} - \frac{1}{9}e^{3t} + \frac{1}{9}.$$

For integration by differentiation, set $f(s) = \int_0^t e^{sx}\,dx$. Then the integral we want to compute is $f'(3)$. Well,

$$f(s) = \frac{1}{s}e^{sx}\Big|_{x=0}^t = \frac{1}{s}\left(e^{st} - 1\right).$$

So

$$f'(s) = \frac{-1}{s^2}\left(e^{st} - 1\right) + \frac{1}{s}\left(te^{st}\right),$$

and

$$f'(3) = -\frac{1}{9}\left(e^{3t} - 1\right) + \frac{1}{3}te^{3t}.$$

(Simple algebra checks that these two answers are the same.)

7. (5 points) Compute $\int_0^1 x^3\sqrt{1-x^2}\,dx$.

Solution: There are three ways to do this one. The most straightforward uses a trigonometric substitution, and appears below. You can also complete the problem with some "ad hoc" substitutions, here either $u = 1 - x^2$ or $u = \sqrt{1-x^2}$ will work.

Let $x = \sin(t)$ (and assume $t \in [-\pi/2, \pi/2]$). Then $\sqrt{1-x^2} = \cos(t)$, $x^3 = \sin^3(t)$, $dx = \cos(t)$, and the bounds on the integral become 0 and $\pi/2$. So we have

$$\int_0^{\pi/2} \sin^3(t)\cos^2(t)\,dt.$$

Since we have an odd power on the sine, we want to save one copy of $\sin(t)$ and take $u = \cos(t)$, turning the rest of the (even number of) sines into cosines by Pythagoras. Explicitly,

$$\begin{aligned}
-\int_0^{\pi/2} (1-\cos^2(t))\cos^2(t)(-\sin(t)\,dt) &= -\int_1^0 (1-u^2)u^2\,du \\
&= \int_0^1 (u^2 - u^4)\,du \\
&= \left[\frac{1}{3}u^3 - \frac{1}{5}u^5\right]_0^1 \\
&= \frac{1}{3} - \frac{1}{5}.
\end{aligned}$$