Solutions

1. (5 points) Find $f^{(6)}(3)$ if the expanion of f in powers of $x-3$ is

$$1-\frac{1}{2}(x-3)^2+37(x-3)^3+\pi(x-3)^4-(x-3)^5+\frac{1}{30}(x-3)^6-81(x-3)^7+\cdots.$$

Solution: By Taylor's formula, we have that the coefficient of $(x-3)^6$ is

$$\frac{f^{(6)}(3)}{6!}=\frac{1}{30}.$$

Solving for the desired quantity gives $f^{(6)}(3)=4!=24$.

It is also valid (but tedious) to actually take the derivatives of the expansion, plugging in $x=3$ after obtaining the 6th derivative. You (of course) get the same answer.

2. (5 points) Define what it means for functions f and g to have *order of contact* k at a point $x=a$.

Solution: It means that $f(a)=g(a)$, $f'(a)=g'(a)$, $f''(a)=g''(a),\ldots,f^{(k)}(a)=g^{(k)}(a)$, but $f^{(k+1)}(a)=g^{(k+1)}(a)$.

3. (5 points) Briefly but carefully describe the consequences of the definition of order of contact in terms of the graphs of the functions.

Solution: When f and g have large order of contact at a point $x=a$, they share a lot of ink nearby $x=a$. Generally, the larger the order of contact, the more ink they share.

4. (5 points) What is $e^{i\pi/6}$, where $i=\sqrt{-1}$?

Solution: Using Euler's formula, we get $e^{i\pi/6}=\cos(\pi/6)+i\sin(\pi/6)=\frac{\sqrt{3}}{2}+\frac{1}{2}i$.

5. (8 points) You are given the expansions

$$\begin{aligned} f(x):&\quad 24(x-1)^3 + 72(x-1)^5 + O(x^6) \\ g(x):&\quad 8(x-1)^3 + 144(x-1)^4 + 16(x-1)^5 + O(x^6). \end{aligned}$$

Use this information to compute $\lim_{x\to 1} \dfrac{f(x)}{g(x)}$.

Solution: We know that we need only consider the first nonzero terms of the expansions of f and g:

$$\lim_{x\to 1} \frac{f(x)}{g(x)} = \lim_{x\to 1} \frac{24(x-1)^3}{8(x-1)^3} = \frac{24}{8} = 3.$$

Note that the second expression here still needs the limit.

6. (10 points) Let $f(x) = \frac{p(x)}{q(x)}$, where p and q have no complex singularities. You are also given that p has zeros at -2 and $1+i$ (and no others), and q has zeros at 3 and $2-i$ (and no others). Find the interval of convergence of the power series of f in powers of $x-1$.

Solution: The function f has no singularities except where $q=0$. Since this occurs at 3 and $2-i$, we need to find the distance from the center of the expansion, 1, to these complex numbers. These distances are

$$\sqrt{(3-1)^2 + (0-0)^2} = 2 \qquad \text{and} \qquad \sqrt{(2-1)^2 + (-1-0)^2} = \sqrt{2}.$$

We need the minimum of these, which is $\sqrt{2}$, as our radius. So the interval of convergence is $(1-\sqrt{2}, 1+\sqrt{2})$.

7. (10 points) What is the harmonic series? Does it converge or not? *Briefly* explain why.

Solution: The harmonic series is

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots + \frac{1}{k} + \cdots .$$

It does not converge. We saw in class that the partial sums

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

grow roughly like $\log(n)$, which goes to infinity as n increases.

8. (10 points) Find a simpler (finite) expression for the number that is

$$1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \cdots + (-1)^k \frac{\pi^{2k}}{(2k)!} + \cdots .$$

Solution: We recognize this as the expansion of $\cos(x)$ with $x = \pi$. Since the expansion for cosine converges for all x, in particular it does so for $x = \pi$, so the above series is equal to $\cos(\pi) = -1$.

9. (13 points) Determine whether the following series converges or not. *Use power series.* Very carefully explain your work.

$$1 + \frac{2}{3!} + \frac{4}{6!} + \frac{8}{9!} + \cdots + \frac{2^k}{(3k)!} + \cdots$$

Solution: We're asked to use power series, so define one. The most natural is

$$f(x) = 1 + \frac{2}{3!}x + \frac{4}{6!}x^2 + \cdots + \frac{2^k}{(3k)!}x^k + \cdots.$$

Since $f(1)$ is the series we were asked about, convergence of the original series is equivalent to the convergence of f on an interval including 1.

We can use either the Ratio Test or the PSCP here. Since most people seem to prefer the Ratio Test, we compute

$$\begin{aligned} \left|\frac{a_{k+1}}{a_k}\right| &= \left|\frac{2^{k+1}R^{k+1}}{(3(k+1))!} \cdot \frac{(3k)!}{2^k R^k}\right| \\ &= \frac{2R}{(3k+3)(3k+2)(3k+1)}. \end{aligned}$$

We need to know for which R is it true that this is less than 1 for large enough values of k. Well, pick any value of R, and you can make k large enough to make $(3k+3)(3k+2)(3k+1) > 2R$; so this is true for all R. Hence the power series f converges for all x, in particular for $x = 1$. So the original series converges.

Note that we have no idea what the original series converges to, and we weren't asked for that.

10. Charlie loves his chocolate bars, but can't stand to see them disappear. Whenever he takes out his chocolate bar, he can only bring himself to break off and eat a third of it, putting the other two-thirds away for the next day.

(a) (3 points) Does Charlie essentially finish off the chocolate bar (in the limit as time $\to \infty$), or is there some untouched portion? (Hint: it might be helpful to consider how much chocolate remains after each snack.)

Solution: At the beginning, Charlie has 1 bar.
After the first day, he has eaten 1/3, so has 2/3 left.
In the second day, he eats 1/3 of what remains, i.e. 1/3 of 2/3. So he has 2/3 of 2/3 left, i.e. $(2/3)^2$.
Continuing in this manner, one can see that after day number k, Charlie has $(2/3)^k$ of his chocolate bar left. In the limit as $k \to \infty$, this amount goes to zero, so he does "essentially" finish off the bar.

(b) (6 points) Charlie's grandpa also loves chocolate, so Charlie decides to skip every other of his days' snacks, giving those portions to Grandpa. If Charlie starts on a fresh chocolate bar, how much of the bar does he end up eating (again, as time $\to \infty$)?

Solution: Charlie is giving up every other day's portion. From the work we did above, in day k there is $(2/3)^k$ of the bar left. So Charlie eats

$$
\begin{aligned}
&\frac{1}{3} + \frac{1}{3}(\text{what remains after day 2}) + \frac{1}{3}(\text{what remains after day 4}) + \cdots \\
=&\frac{1}{3} + \frac{1}{3}\left(\frac{2}{3}\right)^2 + \frac{1}{3}\left(\frac{2}{3}\right)^4 + \cdots \\
=&\frac{1}{3}\left(1 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^4 + \cdots\right) \\
=&\frac{1}{3}\left(1 + x + x^2 + \cdots\right)_{x=4/9} \\
=&\frac{1}{3}\,\frac{1}{1-x}\bigg|_{x=4/9} \\
=&\frac{3}{5}.
\end{aligned}
$$

That is, Charlie eats essentially 3/5 of the chocolate bar. (His Grandpa eats essentially 2/5).

11. Consider the differential equation

$$\cos(x) \cdot y' + \arctan(x) \cdot y = \frac{1}{1-2x}, \qquad y(0) = 7.$$

(a) (15 points) What are the convergence intervals of the power series representations (in powers of x) of

i. $\sin(x)$

Solution: We have this one memorized: $(-\infty, \infty)$.

ii. $\arctan(x)$

Solution: We know $\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$. Since the interval of convergence of $\frac{1}{1+x^2}$ is (-1,1), so is the interval of convergence of $\arctan x$. (Note that $\arctan x$ is NOT $\sin x / \cos x$.)

iii. $\ln(1-x)$

Solution: Again, $\frac{d}{dx} \ln(1-x) = -\frac{1}{1-x}$, and the interval of convergence of the latter function's power series is $(-1, 1)$.

iv. $\dfrac{1}{1-2x}$

Solution: This is the same as $\frac{1}{1-x}$ with $x \mapsto 2x$, so it converges for $2x \in (-1, 1)$, i.e. $x \in (-1/2, 1/2)$.

v. $\sec(x)$

Solution: Secant is defined as $1/\cos(x)$. So it has singularities when $\cos(x) = 0$, which is true at $x = \pm\pi/2, \pm 3\pi/2$, etc. The nearest of these singularities to our expansion point $x = 0$ are $\pm\pi/2$, so the interval of convergence is just $(-\pi/2, \pi/2)$.

(b) (5 points) Using (some of) the above answers, what is the maximum guaranteed interval of convergence for the power series representation of the solution to this differential equation?

Solution: We know that such a power series representation converges on the smallest of the intervals of convergence of the coefficient functions, except for the first coefficient function, which first must be flipped. In this case, we need to consider the functions

$$\frac{1}{\cos(x)}, \qquad \arctan(x), \qquad \frac{1}{1-2x}.$$

From part (a), subparts ii, iv, and v, these have intervals of convergence $(-\pi/2, \pi/2)$, $(-1, 1)$, and $(-1/2, 1/2)$, respectively. The smallest of these is $(-1/2, 1/2)$, and that is our answer.