Solutions

1. (5 points) What is the order of contact of $f(x) = \sin x + \cos x$ and e^x at $x = 0$?

Solution: Compute the derivatives. Any derivative of e^x is e^x, and evaluated at zero this is 1.

$$\begin{array}{rl} f(x) = \sin x + \cos x & f(0) = 1 \\ f'(x) = \cos x - \sin x & f'(0) = 1 \\ f''(x) = -\sin x - \cos x & f''(0) = -1 \end{array}$$

So the first place of disagreement is at the second derivative, hence the functions have order of contact 1 at $x = 0$.

2. (12 points) Compute the following limits:

(a) $\lim_{x\to 5} \frac{f(x)}{g(x)}$, where
$f(5) = 0$, $f'(5) = 0$, $f''(5) = 3$, $f^{(3)}(5) = 1$, $g(5) = 0$, $g'(5) = 0$, $g''(5) = 1$, and $g^{(3)}(5) = 0$.

Solution:

$$= \lim_{x\to 5} \frac{f(5) + f'(5)(x-5) + f''(5)(x-5)^2/2 + \cdots}{g(5) + g'(5)(x-5) + g''(5)(x-5)^2/2\cdots} = \lim_{x\to 5} \frac{3(x-5)^2/2 + O((x-5)^3)}{(x-5)^2/2 + O((x-5)^3)} = 3.$$

(b) $\lim_{x\to 0} \frac{f(x)}{g(x)}$, where $f(0) = 1$, $f'(0) = 0$, $f''(0) = 2$, $g(0) = 0$, $g'(0) = 1$, and $g''(0) = 28$.

Solution:

$$= \lim_{x\to 0} \frac{1 + 2x^2/2 + O(x^3)}{x + 28x^2/2 + O(x^3)} = \lim_{x\to 0} \frac{1}{x} = \pm\infty.$$

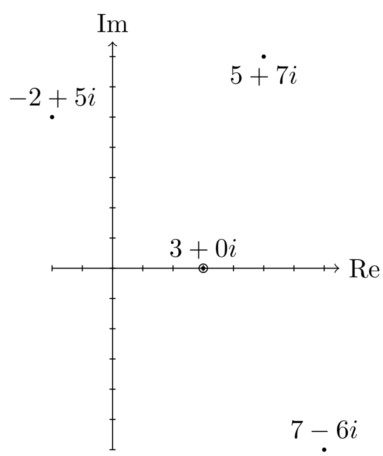
3. (8 points) Suppose you want to estimate $\int_{-10}^{10} e^{-x^2}\,dx$ to within an error of 10^{-6}. How might you do this (using methods from this class)? Be sure to specify how to be certain your estimate is within the desired error bound. (You do not need to write any Mathematica code, just describe the mathematical process.)

Solution: We want to take an expansion of e^{-x^2} out to enough terms so that the error in this estimation is at most $\frac{1}{20}10^{-6}$ for all x between ± 10. If we manage this, then let $P(x)$ be the estimating polynomial. The error in the estimate of the desired integral by the integral of P is given by

$$\begin{aligned}\left|\int_{-10}^{10} e^{-x^2}\,dx - \int_{-10}^{10} P(x)\,dx\right| &= \left|\int_{-10}^{10} \left(e^{-x^2} - P(x)\right)\,dx\right| \\ &\le \int_{-10}^{10} \left|e^{-x^2} - P(x)\right|\,dx \\ &\le \int_{-10}^{10} \frac{1}{20}10^{-6}\,dx \\ &= 10^{-6},\end{aligned}$$

as desired. (Note: you needn't go through this careful proof; a picture can represent the same thing for our purposes.)

4. (9 points) Suppose the function $f(z)$ and its derivatives have complex singularities at $-2+5i$, $5+7i$, and $7-6i$ (and no others). Find the interval of convergence for the expansion in powers of $(x-3)$ of $f(x)$.



Solution:

We need to find the shortest distance from our center, 3+0i, to any of the singularities. The distances are

$$\sqrt{5^2+5^2} = \sqrt{50} \qquad \sqrt{2^2+7^2} = \sqrt{53} \qquad \sqrt{4^2+6^2} = \sqrt{52}$$

so the shortest is $\sqrt{50}$, and the interval of convergence is

$$(3-\sqrt{50}, 3+\sqrt{50}).$$

5. (5 points) Write down the expansion of $\dfrac{1}{1-x}$ in powers of x. What is its interval of convergence?

Solution:

$$1 + x + x^2 + x^3 + \cdots, \qquad \text{for } x \in (-1, 1).$$

6. Using your answers from problem 5, find the expansions in powers of x, and the associated intervals of convergence, of the following functions.

   (a) (5 points) $\dfrac{1}{(1-x)^2}$

   Solution: This function is the derivative of $1/(1-x)$, so we take the derivative of the above series to get

   $$1 + 2x + 3x^2 + 4x^3 + \cdots, \qquad \text{for } x \in (-1, 1).$$

   (b) (5 points) $\dfrac{1}{1-6x}$

   Solution: We just substitute $6x$ into the first expansion to get

   $$1 + (6x) + (6x)^2 + (6x)^3 + \cdots.$$

   The expansion is valid when $6x \in (-1, 1)$, which is equivalent to $x \in (-\frac{1}{6}, \frac{1}{6})$.

7. Determine whether the following series converge. If they do, what do they converge to?

(a) (7 points) $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots+\frac{1}{3^k}+\cdots$

Solution: This is a geometric series; in particular, it is

$$x+x^2+x^3+\cdots+x^k+\cdots$$

where $x=1/3$. So we find the closed form,

$$\begin{aligned} x+x^2+x^3+\cdots &= -1+1+x+x^2+x^3+\cdots \\ &= -1+\frac{1}{1-x} \\ &= -1+\frac{1}{1-\frac{1}{3}} \\ &= \frac{1}{2}. \end{aligned}$$

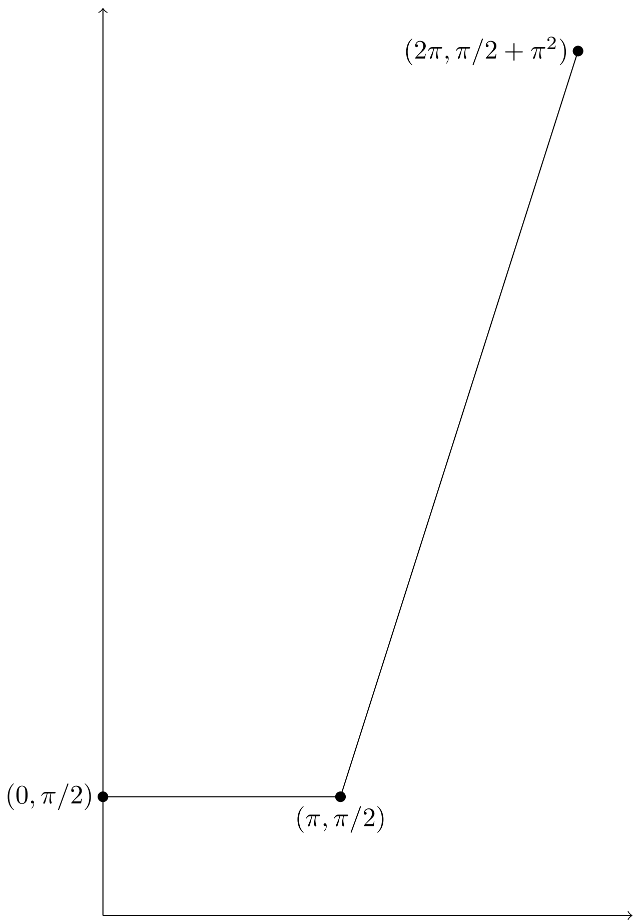
(Note that we made use of the interval of convergence of the geometric series; $\frac{1}{3}\in(-1,1)$.)

(b) (7 points) $\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\cdots+\frac{1}{k}+\cdots$

Solution: This is a tail of the harmonic series, so it diverges.

8. (9 points) Consider the differential equation $y' = x \sin y$, $y(0) = \pi/2$. Use the Euler approximation to fake a plot of the solution to this differential equation on the interval $[0, 2\pi]$, using jump size π.

Solution: Since $y(0) = \pi/2$, $y'(0) = 0 \sin(\pi/2) = 0$. So our first segment below has slope 0. So our next point is $(\pi, \pi/2)$. At this point we get $y'(\pi) \approx \pi \sin(\pi/2) = \pi$, so the next segment has slope π. Hence our final point is $(2\pi, \pi/2 + \pi^2)$.



9. (10 points) Determine the interval of convergence of the power series

$$\frac{1}{3} + \frac{2x^2}{9} + \frac{4x^4}{27} + \cdots + \frac{2^k x^{2k}}{3^{k+1}} + \cdots .$$

Solution: Let's use the Ratio Test. We need to find

$$\frac{2^{k+1} R^{2(k+1)}}{3^{k+2}} \cdot \frac{3^{k+1}}{2^k R^{2k}} = \frac{2R^2}{3},$$

and we want to know for which R is this less than one for all large enough k. Since this doesn't depend on k at all, we just need $R^2 < \frac{3}{2}$, or $|R| < \sqrt{3/2}$. Hence the interval of convergence is $(-\sqrt{3/2}, +\sqrt{3/2})$.

10. (12 points) Determine whether the following series converges or not, using the method from this class; very carefully explain your work.

$$\frac{1}{2} + 8 + \frac{243}{8} + 64 + \cdots + \frac{k^5}{2^k} + \cdots$$

Solution: The method from this class is to first make a power series. Let

$$f(x) = \frac{1}{2}x + 8x^2 + \frac{243}{8}x^3 + 64x^4 + \cdots + \frac{k^5}{2^k}x^k + \cdots .$$

Now we want to find the interval of convergence of this power series, and determine whether or not it includes 1, since $f(1)$ is our desired series.

Let's use the Ratio Test again.

$$\frac{(k+1)^5 R^{k+1}}{2^{k+1}} \cdot \frac{2^k}{k^5 R^k} = \frac{(k+1)^5}{k^5} \cdot \frac{R}{2}.$$

We want to know for which R is this less than one for all large enough k. Well, for very large k, $(k+1)^5/k^5$ is very close to 1, so we ignore that term. We want $R/2 < 1$, i.e. $R < 2$. So the interval of convergence for f is $(-2, 2)$, which does indeed contain $x = 1$, hence the original series converges.

11. Consider the differential equation $(1+x)y' + y = \frac{1}{1-2x}$, $y(0) = 7$.

(a) (6 points) What is the maximum guaranteed interval of convergence for the power series representation of the solution to this differential equation?

Solution: We know that for this kind of differential equation, the power series representation of the solution has a guaranteed interval of convergence that is the smallest among those for

$$\frac{1}{1+x}, \qquad 1, \qquad \frac{1}{1-2x}.$$

The intervals of convergence of these are

$$(-1, 1), \qquad (-\infty, \infty), \qquad (-\frac{1}{2}, \frac{1}{2}),$$

respectively, so the maximum guaranteed interval of convergence is $(-\frac{1}{2}, \frac{1}{2})$.

(b) Bonus (only if you have time): Find the first three or four terms of the power series representation of the solution to the differential equation.

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