Name: ______________________________

- **READ THE FOLLOWING DIRECTIONS!**
- **Do NOT open the exam until instructed to do so.**
- You have until 10:00pm to complete this exam. When you are told to stop writing, do it or you will lose all points on the page you write on.
- You may not communicate with other students during this test.
- No written materials of any kind are allowed. No scratch paper is allowed except as given by the proctors.
- No phones, calculators, or any other electronic devices are allowed for any reason, including checking the time (a simple wristwatch is fine).
- Any case of cheating will be taken extremely seriously.
- Show all your work and explain your answers.
- Before turning in your exam, check to make certain you've answered all the questions.
- You do not need to simplify algebraic expressions.
- Be sure to check whether a problem asks you to *compute* or just *set up* an integral.
- When you apply a theorem, say so.

Some possibly useful formulas:

$$\cos^2 t = \frac{1}{2}(1 + \cos(2t))$$
$$\sin^2 t = \frac{1}{2}(1 - \cos(2t))$$

I have not had the time to compile a long list of questions for you to practice. Some resources:

- Spring 2013's final exam (my webpage, teaching archive)
- Spring 2013's final exam practice
- this and last semester's hourly exams and practice
- quizzes and homework (written and electronic)
- Literacy files, Basics and Tutorials
- Calc textbooks (library, lab)
- traditional Calc 3 material? (e.g., `www.math.uiuc.edu/~clein/classes/2013/fall/final.html`)

In addition I have a few problems that follow here. First, a brief outline of topics (see also the "content" files for each of the hourly exams):

We need to start by familiarizing ourselves with vectors and geometry of 2D and 3D spaces. In particular, we studied vector operations (addition, scalar multiplication, dot and cross products, projections, angles between vectors) and lines and planes (including how they interact, with parallel/perpendicular/neither and intersecting/not).

Then we want to extend your usual notion of real-valued one-variable functions. There are two places to increase dimension: in the input or the output.

To have higher-dimensional output, we study vector functions (still of one variable). These can be thought of as curves in 2- or 3-dimensional spaces, and frequently we think of the motion of a particle. We can define derivatives just as in calc 1 here, and the derivative is the velocity vector for the particle's motion, which is always tangent to the curve. The second derivative is the acceleration vector, which we sometimes break into its tangential and normal components. In 2D it is then easy to find the normal vector as well ($\langle dy, -dx\rangle$). In 3D it take some more work, taking the derivative of the unit tangent vector; and from there we can find a binormal vector using the cross product.

If instead we take higher-dimensional input, we get multivariate functions (still real-valued). We visualize these both with graphs and contour maps. We can take partial derivatives, which lead to the gradient vector, tangent planes, and optimization. We can integrate these functions over curves, 2D regions and surfaces, and 3D regions. This allows us to compute probabilities, average values and centroids, areas and volumes, and masses. We sometimes employ transformations to make these easier.

If we increase dimensions in both input and output equally, we get vector fields. Here we care about trajectories, flow along curves, flow across curves (in 2D) and surfaces (in 3D), sources and sinks and swirls. Gradient fields are especially nice vector fields. Singularities give us some trouble, but are interesting and important. The divergence, rotation (2D), and curl (3D) of vector fields come in handy.

Then finally we have several relationships between these concepts, in the form of big-name theorems: the fundamental theorem of path integrals, Gauss-Green theorem, divergence theorem, and Stokes's theorem.

1. Let $f(x, y) = 4xy - x^4 - y^4$. Find and classify all local extrema of $f$. Sketch a contour map of $f$ based on this information.

2. Compute the net flow of $\vec{F}(x,y,z) = \left\langle \arctan\dfrac{x}{y},\, \ln\sqrt{x^2+y^2},\, 1 \right\rangle$ along the triangle $C$ with vertices $(0,0,0)$, $(1,1,1)$, and $(0,0,2)$.

3. (a) Suppose there is a curve $C$ in $\mathbb{R}^3$ that is the boundary of some surface $R$, and that the vector field $\vec{F}$ has the peculiar property that $\operatorname{curl} \vec{F}$ is a unit normal vector for $R$ at every point. Find the net amount of flow of $\vec{F}$ along $C$ in terms of some geometric measurement(s). (Don't worry about the direction.)

   (b) What if $\operatorname{curl} \vec{F}$ is instead a unit tangent vector to $R$ at every point?

4. Consider the vector field $\vec{F}(x, y, z) = \langle 1 - xz + \frac{1}{2}z^2, 1 - yz, xz \rangle$. Find the flow of $\vec{F}$ along the unit circle in the $xy$-plane by...

   (a) ...directly computing a path integral.

   (b) ...using Gauss-Green on the 2D field $\vec{F_0} = \langle y, 1 - x \rangle$ in the $xy$-plane. (This field is the restriction of $\vec{F}$ to the $xy$-plane.)

   (c) ...using Stokes's formula and the disk in the $xy$-plane (with $z = 0$).

   (d) ...using Stokes's formula and the hemisphere $z = \sqrt{1 - x^2 - y^2}$.

   (e) ...using Stokes's formula and the paraboloid $z = 1 - x^2 - y^2$.

   (Check out problem 3 and how it relates here.)