Name: Solutions

- **READ THE FOLLOWING DIRECTIONS!**
- **Do NOT open the exam until instructed to do so.**
- You have until 12:50 to complete this exam. When you are told to stop writing, do it or you will lose all points on the page you write on.
- You may not communicate with other students during this test.
- No written materials of any kind are allowed. No scratch paper is allowed except as given by the proctors.
- No phones, calculators, or any other electronic devices are allowed for any reason, including checking the time (a simple wristwatch is fine).
- Any case of cheating will be taken extremely seriously.
- Show all your work and explain your answers.
- Before turning in your exam, check to make certain you've answered all the questions.
- You do not need to simplify algebraic expressions.
- Be sure to check whether a problem asks you to *compute* or just *set up* an integral.
- When you apply a theorem, say so.

Some possibly useful formulas:

$$\cos^2 t = \frac{1}{2}(1+\cos(2t))$$
$$\sin^2 t = \frac{1}{2}(1-\cos(2t))$$

Question:	1	2	3	4	5	6	7	Total
Points:	16	12	24	16	12	4	16	100
Score:								

1. (16 points) Let $\vec{F}(x,y,z) = \langle x^2e^y, 2e^y - y^2z, yz^2\rangle$, and let R be the sphere of radius 2 centered at $(4,2,1)$. Is the net flow of $\vec{F}$ across R inward or outward? (Note that you are not asked to find the amount of flow, just the direction.)

$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \partial_x(x^2e^y) + \partial_y(2e^y - y^2z) + \partial_z(yz^2)$$
$$= 2xe^y + 2e^y - 2yz + 2yz$$
$$= 2(x+1)e^y \quad \begin{array}{l} >0 \iff x>-1 \\ <0 \iff x<-1 \\ =0 \iff x=-1 \end{array}$$

Inside R, $x \geq 4-2=2$,

so inside R $\operatorname{div}\vec{F} > 0$. So

$$\text{flow across } R = \iint_R \vec{F}\cdot \overset{\text{outward}}{d\vec{S}} \underset{\substack{\text{Divergence}\\ \text{Theorem}}}{=} \iiint_{\substack{\text{inside}\\ R}} \operatorname{div}\vec{F}\, dV > 0,$$

so flow is outward.

2. (12 points) Find the volume of the region bounded by the surfaces

$$x+y=0, \qquad x+y=5,$$
$$x-y=2, \qquad x-y=3,$$
$$x+z=y^2, \text{ and } \qquad x+z=y^2+4.$$

Let $u = x+y$ so $u \in [0,5]$
$v = x-y$ $\quad v \in [2,3]$
$w = x-y^2+z$ $\quad w \in [0,4]$.

$$\frac{1}{J} = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{array}{c|ccc} & x & y & z \\ \hline u & 1 & 1 & 0 \\ v & 1 & -1 & 0 \\ w & 1 & -2y & 1 \end{array} = 0 - 0 + 1 \cdot \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$$

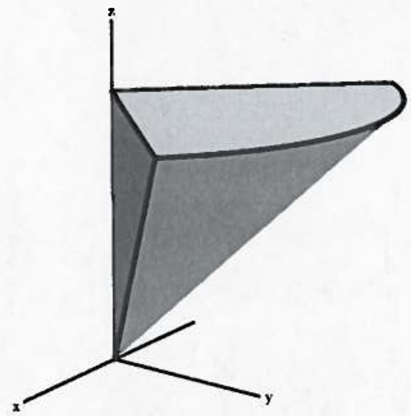
$$\Rightarrow |J| = \frac{1}{2}.$$

$$\text{Volume} = \iiint_{\substack{\text{inside}\\\text{region}}} 1 \, dx\,dy\,dz = \int_0^4 \int_2^3 \int_0^5 1 \cdot \frac{1}{2} \, du\,dv\,dw$$

$$= \frac{1}{2} \cdot \text{Vol}\begin{pmatrix} \text{rectangular box} \\ u \in [0,5],\ v \in [2,3], \\ w \in [0,4] \end{pmatrix}$$

$$= \frac{1}{2} \cdot 5 \cdot 1 \cdot 4 = 10.$$

3. (24 points)



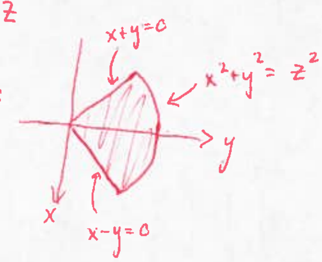
Consider the region shown to the right; it is bounded by the cone $z = \sqrt{x^2+y^2}$ and the planes $x+y=0$, $x-y=0$, and $z=4$, and consists of points with $y \geq 0$.
Set up $\iiint_E y\,dV$ with *two of the following three methods*: slices, sticks, and spherical coordinates. Be sure to clearly label your approaches. You may attempt all three, but clearly label which two you want graded. (The third may be worth extra credit.)

slices: (fixed z here)

$$\int_0^4 \left(\iint_{R(z)} y \, dx\,dy \right) dz$$

$$= \int_0^4 \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_0^z (r\sin\theta)(r)\,dr\,d\theta\,dz$$

R(z):

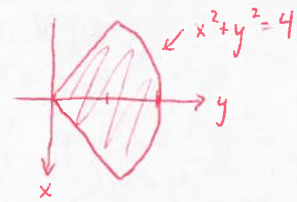


sticks: (z direction here)

$$\iint_R \left(\int_{\sqrt{x^2+y^2}}^4 y \, dz \right) dx\,dy$$

$$= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_0^2 \left(\int_r^4 (r\sin\theta)\,dz \right) r\,dr\,d\theta$$

R:



spherical:

$$\int_{\pi/4}^{3\pi/4} \int_0^{\pi/4} \int_0^{4\sec\varphi} (\rho\sin\varphi\sin\theta)(\rho^2\sin\varphi)\,d\rho\,d\varphi\,d\theta$$

4. Consider the vector field $\vec{F}(x,y,z) = \langle 3x^2y, x^3+e^z, ye^z\rangle$.

(a) (8 points) Verify that $\vec{F}$ is a gradient field. (Show enough work for me to know that you know what you're doing.)

$$\text{curl}\,\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ 3x^2y & x^3+e^z & ye^z \end{vmatrix} = \langle e^z - e^z,\ -0+0,\ 3x^2-3x^2\rangle$$

$$= \langle 0,0,0\rangle$$

$\Rightarrow \vec{F}$ is a gradient field.

(b) (8 points) What is the flow of $\vec{F}$ along the curve $\underbrace{\left\langle 27+\cos t, 43-\sin t, e^{84t^2}\sin t\right\rangle}_{\ell(t)=}$, $t\in[0,2\pi]$?

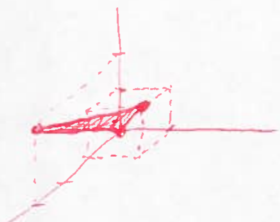
$\ell(0) = \langle 28, 43, 0\rangle = \ell(2\pi)$. $\boxed{0}$

Net flow of a gradient field along a closed curve is zero.
(By Stokes's Theorem or Fundamental Theorem of Path Integrals.)

5. (12 points) Use Stokes's Theorem to compute the net amount of flow of $\vec{F}(x,y,z) = \langle 2y, 3z, x\rangle$ along the triangle C with vertices $(0,0,0)$, $(1,1,1)$, and $(3,0,2)$. (Note that you are not asked to find the direction.)

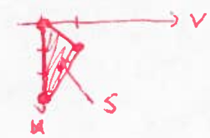
(Hint: the plane passing through these three points is $2x + y - 3z = 0$. Your final integral should be easy based on some geometry.)

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ 2y & 3z & x \end{vmatrix} = \langle -3, -1, -2\rangle.$$



R = solid triangle on plane $2x+y-3z=0$ inside C.

Parametrize R: $x = u$, $y = v$, $z = \frac{1}{3}(2u+v)$



$$\vec{dS} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2/3 \\ 0 & 1 & 1/3 \end{vmatrix} = \langle -\tfrac{2}{3}, -\tfrac{1}{3}, 1\rangle.$$

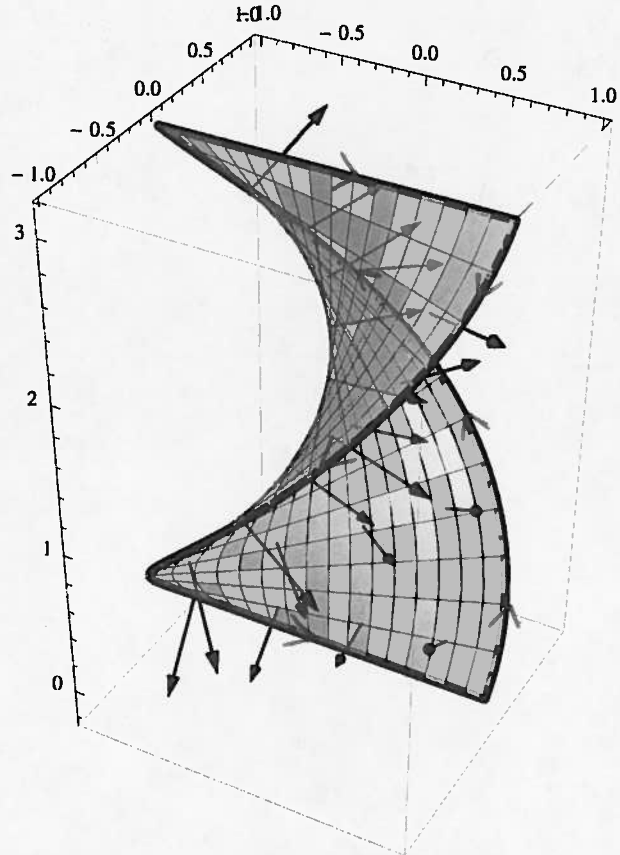
$$\text{flow along } C = \int_C \vec{F}\cdot\vec{dr} \underset{\text{stokes}}{=} \iint_R \text{curl}\,\vec{F}\cdot\vec{dS}$$

$$= \iint_S \langle -3,-1,-2\rangle \cdot \langle -\tfrac{2}{3}, -\tfrac{1}{3}, 1\rangle\, du\,dv$$

$$= \iint_S \tfrac{1}{3}\, du\,dv$$

$$= \tfrac{1}{3}\,\text{Area}(S) = \tfrac{1}{3}\cdot\tfrac{1}{2}\cdot 3\cdot 1 = \boxed{\tfrac{1}{2}}.$$

6. (4 points) Below is a surface with a selection of normal direction. Indicate on the picture the orientation of the boundary of the surface that "matches" in the sense of Stokes's Theorem. (To be clear: the normals shown point toward you near the bottom of the picture, and away from you near the top of the picture.)



7. (16 points) Let R be the surface with parametrization $\langle u^2, uv, v\rangle$, with $u \in [-1,1]$ and $v \in [0,1]$. Suppose a sheet of metal in the shape of R has density xy^2 at each point. Set up a double integral in u, v that measures the mass of this sheet. (The integral should be over a region in the uv-plane, with no vectors involved.)

$$\text{Mass} = \iint_R xy^2\, dS$$

$$= \int_0^1 \int_{-1}^1 (u^2)(uv)^2 \sqrt{v^2+4u^2+4u^4}\, du\, dv$$

$$\vec{dS} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2u & v & 0 \\ 0 & u & 1 \end{vmatrix} = \langle v, -2u, 2u^2 \rangle.$$

$$dS = \|\vec{dS}\| = \sqrt{v^2+4u^2+4u^4}.$$