Name: Solutions

- **READ THE FOLLOWING DIRECTIONS!**
- **Do NOT open the exam until instructed to do so.**
- You have until 12:50 to complete this exam. When you are told to stop writing, do it or you will lose all points on the page you write on.
- You may not communicate with other students during this test.
- No written materials of any kind are allowed. No scratch paper is allowed except as given by the proctors.
- No phones, calculators, or any other electronic devices are allowed for any reason, including checking the time (a simple wristwatch is fine).
- Any case of cheating will be taken extremely seriously.
- Sign below these instructions to indicate that you have read and agree to them.
- Show all your work and explain your answers.
- Before turning in your exam, check to make certain you've answered all the questions.
- You do not need to simplify algebraic expressions.

Some possibly useful formulas:

$$\cos^2 t = \frac{1}{2}(1 + \cos(2t))$$
$$\sin^2 t = \frac{1}{2}(1 - \cos(2t))$$

Question:	1	2	3	4	5	6	7	8	Total
Points:	6	12	24	18	12	8	10	10	100
Score:									

1. (6 points) Give the formulas for the centroid of a region R in the plane.

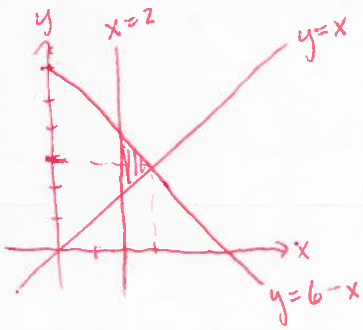
$(\bar{x}, \bar{y})$, where

$$\bar{x} := \frac{1}{\text{Area}(R)} \iint_R x \, dA$$

$$\bar{y} := \frac{1}{\text{Area}(R)} \iint_R y \, dA$$

(and $\text{Area}(R) := \iint_R 1 \, dA$).

2. (12 points) Compute $\iint_R x^2 \, dA$, where R is the region in the plane bounded by the lines $y = x$, $x = 2$, and $y = 6 - x$.



$$= \int_2^3 \int_x^{6-x} x^2 \, dy \, dx$$

$$= \int_2^3 x^2 y \Big|_{y=x}^{6-x} dx$$

$$= \int_2^3 \left(x^2(6-x) - x^3\right) dx$$

$$= \int_2^3 \left(6x^2 - 2x^3\right) dx$$

$$= \left[2x^3 - \tfrac{1}{2}x^4\right]_2^3$$

$$= \left(54 - \tfrac{81}{2}\right) - (16 - 8)$$

$$\left(= \tfrac{11}{2}\right).$$

3. Consider the vector field $\vec{F}(x,y) = \left\langle \dfrac{x-1}{(x-1)^2+y^2}, \dfrac{y}{(x-1)^2+y^2} \right\rangle$.

(a) (6 points) Which part(s) of the gradient test does $\vec{F}$ pass?

$\vec{F}$ has a singularity at $(1,0)$ ✗

$\operatorname{curl} \vec{F} = \partial_x\left(\frac{y}{(x-1)^2+y^2}\right) - \partial_y\left(\frac{x-1}{(x-1)^2+y^2}\right) = \frac{-2(x-1)y}{((x-1)^2+y^2)^2} - \frac{-2(x-1)y}{((x-1)^2+y^2)^2} = 0$ ✓

(b) (10 points) Directly compute the flow of $\vec{F}$ along C_3 shown below.

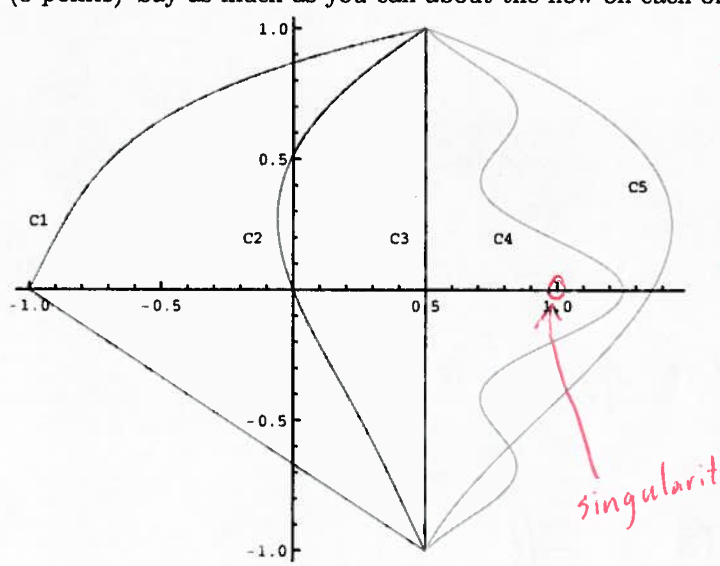
Parametrize C_3: $x=0.5$, $y=t$, $t\in[-1,1]$

flow along $= \int_{C_3} \vec{F}\cdot\langle dx, dy\rangle = \int_{-1}^{1} \left\langle \cdots, \frac{t}{\frac{1}{4}+t^2}\right\rangle \cdot \langle 0,1\rangle\, dt$

$= \int_{-1}^{1} \frac{t}{\frac{1}{4}+t^2}\, dt = 0.$

odd function

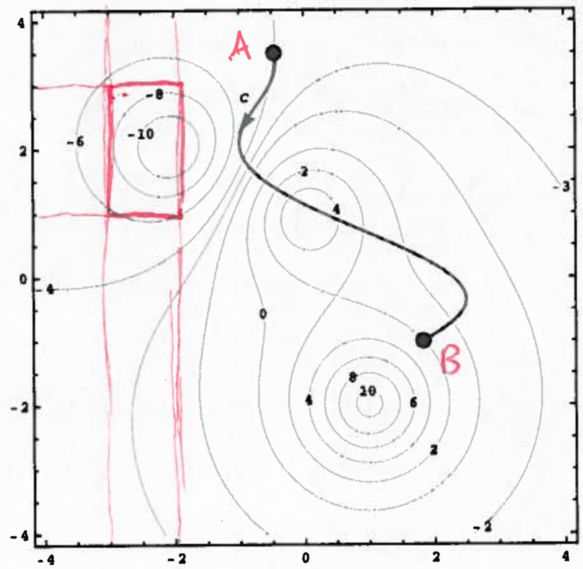
(c) (8 points) Say as much as you can about the flow on each of the other curves shown.



Flow along C_1 & C_2 is same as C_3, i.e. zero.

Flow along C_4 is same as C_5, but can't (easily) tell what this flow is.

4. Below is shown the contour map of a function f, together with a curve C.



(a) (8 points) Find $\int_C \nabla f \cdot \langle dx, dy \rangle$. (Note that the orientation of C is given in the picture.)

$= f(B) - f(A) = 2 - (-4) = 6.$

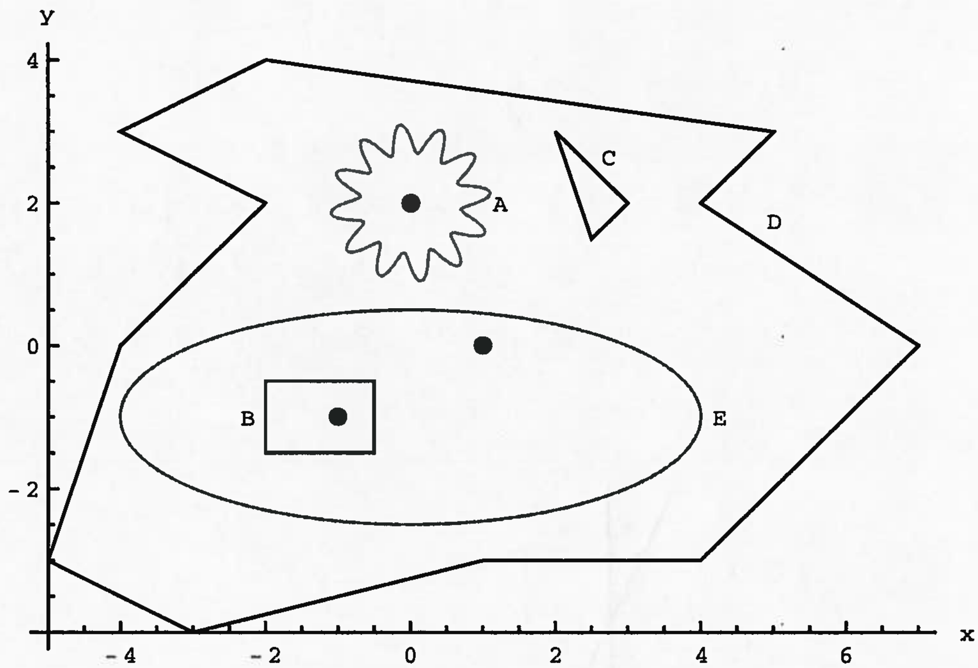
(b) (10 points) Estimate $\iint_R f\, dA$, where R is the solid rectangle with vertices $(-3, 1), (-3, 3), (-2, 3), (-2, 1)$.

Average value appears to be ≈ -8,

and area $= 2$, so

$\iint_R f\, dA \approx -16.$

5. (12 points) A certain vector field $\vec{F}$ has $\operatorname{rot}\vec{F} = 1$ and $\operatorname{div}\vec{F} = 0$ everywhere except at its three singularities at $(-1,-1)$, $(0,2)$, and $(1,0)$. Below are shown several curves as well as the three singularities of $\vec{F}$.



You are given the following pieces of information:

$$\int_A \vec{F}\cdot\langle dy, -dx\rangle = 4, \qquad \int_D \vec{F}\cdot\langle dy, -dx\rangle = 7.$$

(Assume all curves are parametrized counterclockwise.) Say as much as possible about the following. Give brief explanations.

(a) net flow of $\vec{F}$ along C $=$ Area(int C) $\cdot$ 1 , counterclockwise

no singularities $= \iint_{\text{int } C} \operatorname{rot}\vec{F}\, dA$

(b) net flow of $\vec{F}$ across C

$= \iint_{\text{int } C} \operatorname{div}\vec{F}\, dA = 0$

(c) net flow of $\vec{F}$ across E

$=$ Net flow across D $-$ Net flow across A $= 7-4=3$
outward

(d) net flow of $\vec{F}$ across B

cannot be determined

6. (8 points) Find all sources and sinks of the vector field $\vec{F}(x,y) = \langle x^4, x^2y^2 \rangle$

$$\operatorname{div}\vec{F} = \partial_x(x^4) + \partial_y(x^2y^2)$$

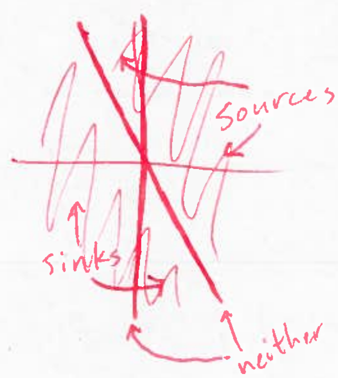
$$= 4x^3 + 2x^2y$$

$$= 2x^2(2x+y)$$

$>0 \iff 2x+y>0$ sources

$<0 \iff 2x+y<0$ sinks

$=0 \iff 2x+y=0$ or $x=0$ neither



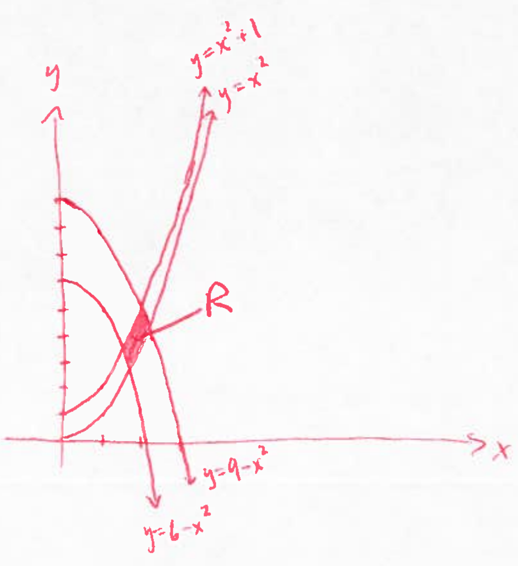
7. (10 points) Compute the area of the region R in the first quadrant bounded by $y = x^2$, $y = x^2 + 1$, $y = 6 - x^2$, and $y = 9 - x^2$. You may stop when you have a double integral over a rectangle.

Let $u = y - x^2$
$v = y + x^2$. Then $y = \frac{1}{2}(u+v)$,
$x = +\sqrt{\frac{1}{2}(v-u)}$.

region is in 1^{st} quadrant

$$J = \begin{vmatrix} \partial_u x & \partial_v x \\ \partial_u y & \partial_v y \end{vmatrix} = \begin{vmatrix} \frac{-1}{4\sqrt{\frac{1}{2}(v-u)}} & \frac{1}{4\sqrt{\frac{1}{2}(v-u)}} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\frac{1}{4\sqrt{\frac{1}{2}(v-u)}}$$

$$\text{Area} = \iint_R 1 \, dx\,dy = \int_6^9\int_0^1 \frac{1}{4\sqrt{\frac{1}{2}(v-u)}} \, du\,dv.$$



8. (10 points) Compute $\iint_D e^{x^2+y^2}\,dx\,dy$, where D is the quarter of the disk of radius 2 that lies in the first quadrant. That is, D is defined by the inequalities $x^2+y^2 \le 4$, $x \ge 0$, and $y \ge 0$.

$$= \int_0^{\pi/2} \int_0^{2} e^{r^2}\, r\, dr\, d\theta \qquad u = r^2,\quad du = 2r\,dr$$

$$= \frac{1}{2} \int_0^{\pi/2} \int_0^{4} e^{u}\, du\, d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} (e^4 - 1)\, d\theta = \frac{\pi}{4}(e^4 - 1).$$