Name: Solutions

- **READ THE FOLLOWING DIRECTIONS!**
- **Do NOT open the exam until instructed to do so.**
- You have until 12:50 to complete this exam. When you are told to stop writing, do it or you will lose all points on the page you write on.
- You may not communicate with other students during this test.
- No written materials of any kind are allowed. No scratch paper is allowed except as given by the proctors.
- No phones, calculators, or any other electronic devices are allowed for any reason, including checking the time (a simple wristwatch is fine).
- Any case of cheating will be taken extremely seriously.
- Show all your work and explain your answers.
- Before turning in your exam, check to make certain you've answered all the questions.
- You do not need to simplify algebraic expressions.

Question:	1	2	3	4	5	6	7	Total
Points:	17	15	22	6	10	14	16	100
Score:								

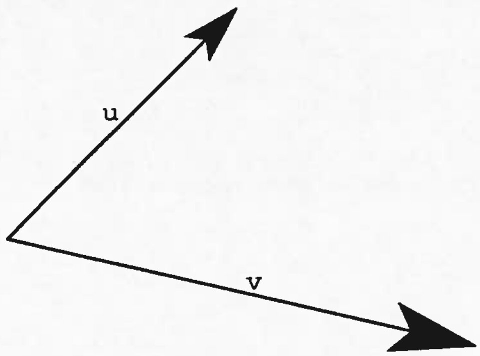
1. Quickies:

(a) (5 points) Parametrize the line segment joining $(1, 3, 2)$ to $(-1, 2, 4)$.

$\ell(t) = (1-t)\langle 1,3,2\rangle + t\langle -1,2,4\rangle, \quad t \in [0,1]$

$= \langle 1,3,2\rangle + t\langle -2,-1,2\rangle$

(b) (4 points) Here are two vectors $\vec{u}$ and $\vec{v}$ living in the plane of this paper. Describe the direction of $\vec{u} \times \vec{v}$.



into the page

by right hand rule

(c) (3 points) Describe the magnitude of the same $\vec{u} \times \vec{v}$ in terms of some geometry.

= area of parallelogram spanned by $\vec{u}, \vec{v}$:

(d) (4 points) Is the cross product commutative? That is, does $\vec{a} \times \vec{b} = \vec{b} \times \vec{a}$ for every $\vec{a}, \vec{b}$? Explain briefly.

No: $a \times b = -b \times a$

(e) (1 point) Is the cross product associative? That is, does $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c}$ for all $\vec{a}, \vec{b}. \vec{c}$? Explain briefly.

No: for example, $(\hat{\imath} \times \hat{\imath}) \times \hat{\jmath} = \vec{0} \times \hat{\jmath} = \vec{0}$

but $\hat{\imath} \times (\hat{\imath} \times \hat{\jmath}) = \hat{\imath} \times \hat{k} = -\hat{\jmath}$

2. Consider the planes $x - y + 2z = 4$ and $2x - 3y + z = 6$.

(a) (3 points) Explain why, at a glance, you know these planes are not parallel.

$\langle 1, -1, 2 \rangle \neq c \langle 2, -3, 1 \rangle$ for any c.

(b) (6 points) Find a vector that is parallel to both planes.

$$\langle 1, -1, 2 \rangle \times \langle 2, -3, 1 \rangle$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & 2 \\ 2 & -3 & 1 \end{vmatrix}$$

$$= \langle -1+6, -(1-4), -3+2 \rangle$$

$$= \langle 5, 3, -1 \rangle$$

(c) (6 points) Give an equation for the line that is the intersection of the two planes.

$$\ell(t) = \langle \tfrac{8}{3}, 0, \tfrac{2}{3} \rangle + t \langle 5, 3, -1 \rangle.$$

$$\begin{cases} x - y + 2z = 4 \\ 2x - 3y + z = 6 \end{cases}$$

$$y = 0 \implies \begin{cases} x + 2z = 4 & ① \\ 2x + z = 6 & ② \end{cases}$$

$$② - 2 \cdot ① : \quad -3z = -2$$

$$z = 2/3$$

$$x = 8/3$$

3. Suppose a particle moves in space, with position $(\sin t,\ \cos t,\ t - t^2)$ at time t.

(a) (4 points) Find the velocity at time $t = \pi/2$.

$v(t) = \langle \cos t, -\sin t, 1 - 2t \rangle$

$v(\frac{\pi}{2}) = \langle 0, -1, 1 - \pi \rangle$

(b) (4 points) Find the acceleration at time $t = \pi/2$.

$a(t) = \langle -\sin t, -\cos t, -2 \rangle$

$a(\frac{\pi}{2}) = \langle -1, 0, -2 \rangle$

(c) (6 points) Find the tangential component of acceleration at time $t = \pi/2$.

$= \text{proj}_{v(\frac{\pi}{2})} a(\frac{\pi}{2}) = \dfrac{a \cdot v}{v \cdot v} v = \dfrac{0 + 0 - 2(1-\pi)}{0 + 1 + (1-\pi)^2} \langle 0, -1, 1-\pi \rangle$

$= \dfrac{2(\pi - 1)}{(\pi - 1)^2 + 1} \langle 0, -1, 1 - \pi \rangle.$

(d) (4 points) Find the normal component of acceleration at time $t = \pi/2$.

$= a(\frac{\pi}{2}) - \text{proj}_{v(\frac{\pi}{2})} a(\frac{\pi}{2}) = \langle -1, 0, -2 \rangle - \dfrac{2(\pi-1)}{(\pi-1)^2+1} \langle 0, -1, 1-\pi \rangle$

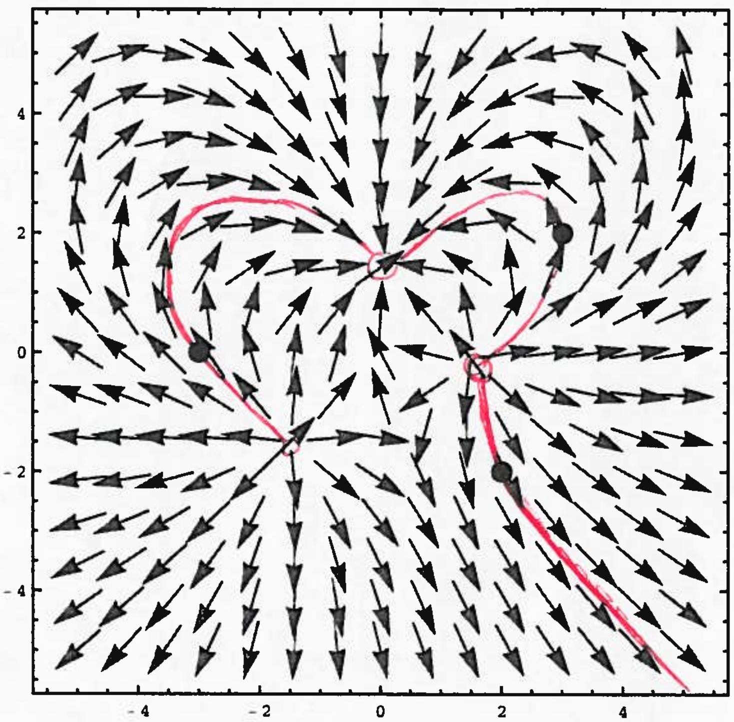
$\left(= \left\langle -1, \dfrac{2(\pi-1)}{(\pi-1)^2+1}, \dfrac{1}{(\pi-1)^2+1} \right\rangle \right).$

(e) (4 points) What do you know about how the speed of the particle is changing at $t = \pi/2$? How do you know?

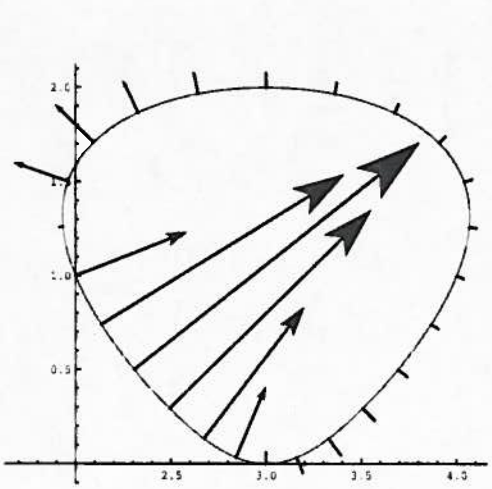
From (c), since $a \cdot v = 2(\pi - 1) > 0$,

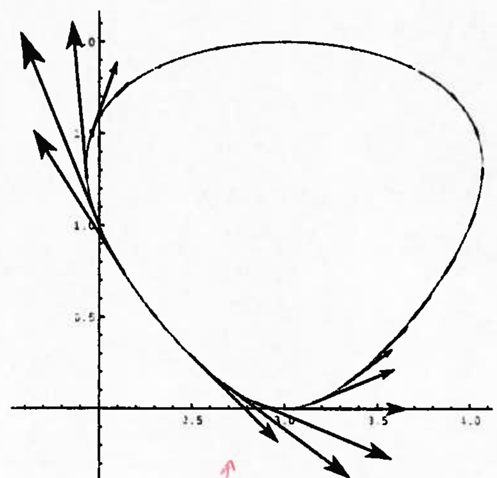
the particle's speed is increasing.

4. (6 points) Here's a (scaled) plot of a certain vector field $\vec{F}(x, y)$. Throw in the trajectories that pass through the indicated points.



5. (10 points) Now we'll just look at $\vec{F}$ on a curve C. Shown below are the tangential and normal components of $\vec{F}$ on C. What do each of them tell you about the net flow of $\vec{F}$ along/across C?





This indicates that the net flow is <u>inward</u> (more vectors go outward, but those going inward are much longer).

This seems to indicate that the net flow <u>along</u> is zero, or at least close to zero; clockwise & counterclockwise contributions are roughly balanced.

6. (14 points) Find the maximum and minimum values of $f(x, y) = x^3y$ on the disk $x^2 + y^2 \leq 4$. Then sketch the region together with the level curves for f corresponding to your maximum and minimum.

Interior: $\nabla f = \langle 3x^2y, x^3 \rangle = \langle 0, 0 \rangle$

$\Leftrightarrow x = 0$, $\begin{pmatrix} y \in \mathbb{R} \\ \text{or} \\ y \in (-2,2) \end{pmatrix}$ $\Big\}$ $f = \boxed{0}$

Boundary: $g(x,y) = x^2 + y^2$

use Lagrange multipliers:

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 4 \end{cases} \Leftrightarrow \begin{cases} 3x^2y = 2\lambda x & (1) \\ x^3 = 2\lambda y & (2) \\ x^2 + y^2 = 4 & (3) \end{cases}$$

If $x, y \neq 0$, then $\frac{3x^2y}{x} \overset{(1)}{=} 2\lambda \overset{(2)}{=} \frac{x^3}{y}$

$\Rightarrow 3y^2 = x^2$ (4)

$\overset{(3)}{\Rightarrow} 3y^2 + y^2 = 4 \Rightarrow y = \pm 1$

$\overset{(4)}{\Rightarrow} x = \pm\sqrt{3}$

If $x = 0$ or $y = 0$, then $f = \boxed{0}$

$f(-\sqrt{3}, -1) = f(\sqrt{3}, 1) = \boxed{\sqrt{27}}$

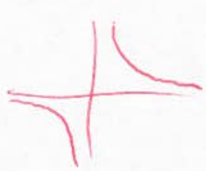
$f(-\sqrt{3}, 1) = f(\sqrt{3}, -1) = \boxed{-\sqrt{27}}$

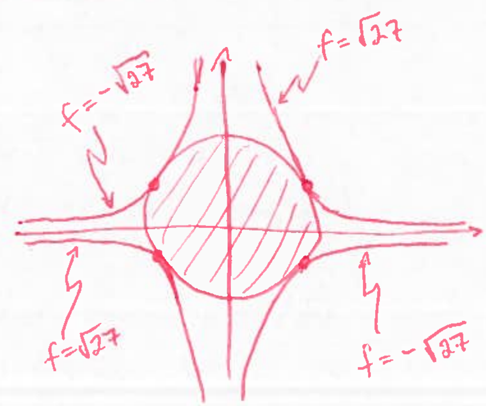
$\min f = -\sqrt{27}$

$\max f = +\sqrt{27}$

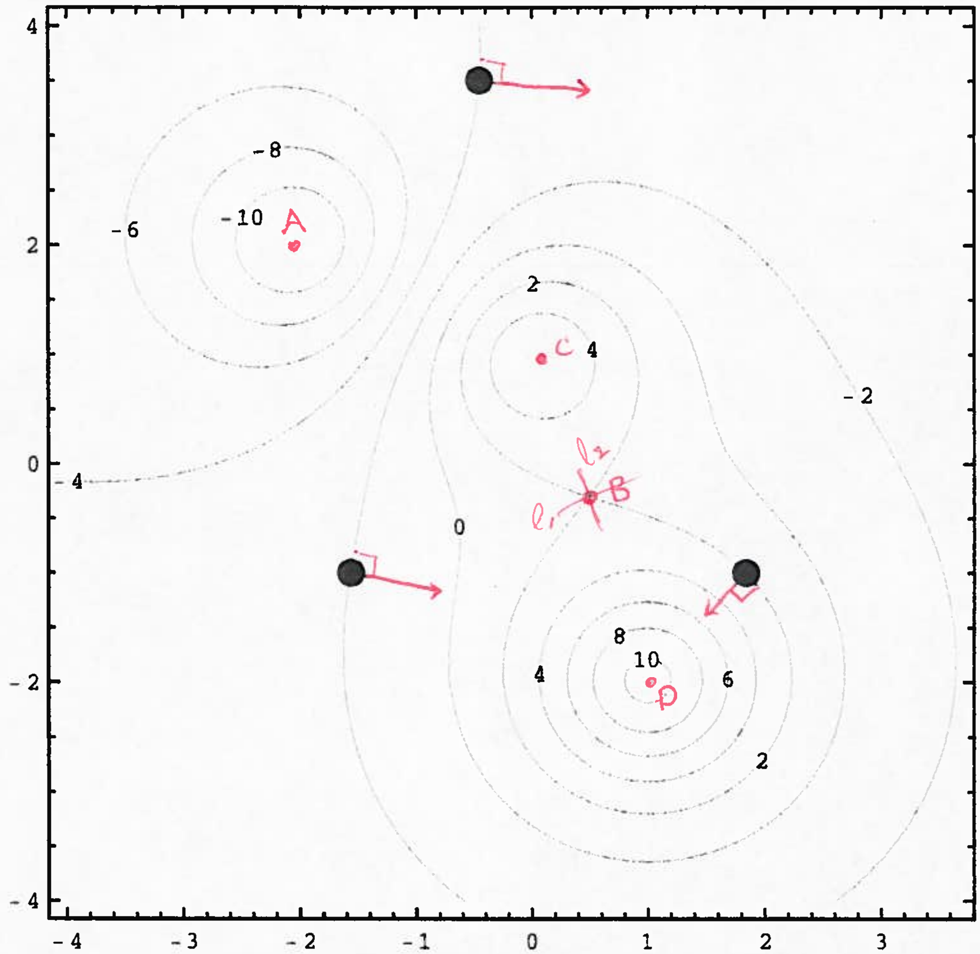
$x^3y = \sqrt{27}$

$\Leftrightarrow y = \frac{\sqrt{27}}{x^3}$:





7. Below is a plot of several level curves of a function $f(x, y)$ inside the rectangle R.



(a) (9 points) At the indicated points, sketch in the gradient vectors. (not to scale; in direction of greatest increas

(b) (7 points) Mark the (approximate) locations of any critical points of f in R, and classify them as local max, min, or saddle points.

A – local min
C, D – local max
centers of concentric level curves – A decreasing values, C&D increasing values

B – saddle: max in direction ℓ_1, min in direction ℓ_2

Scratch Paper - Do Not Remove