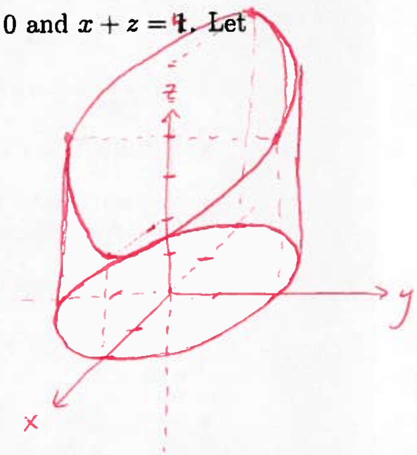
Name: Solutions

- **READ THE FOLLOWING DIRECTIONS!**
- **Do NOT open the exam until instructed to do so.**
- You have until 12:50 to complete this exam. When you are told to stop writing, do it or you will lose all points on the page you write on.
- You may not communicate with other students during this test.
- No written materials of any kind are allowed. No scratch paper is allowed except as given by the proctors.
- No phones, calculators, or any other electronic devices are allowed for any reason, including checking the time (a simple wristwatch is fine).
- Any case of cheating will be taken extremely seriously.
- Show all your work and explain your answers.
- Before turning in your exam, check to make certain you've answered all the questions.
- You do not need to simplify algebraic expressions.

Some possibly useful formulas:

$$\cos^2 t = \frac{1}{2}(1 + \cos(2t))$$
$$\sin^2 t = \frac{1}{2}(1 - \cos(2t))$$

1. Let E be the 3D region bounded by the cylinder $x^2 + y^2 = 4$ and the planes $z = 0$ and $x + z = 4$. Let R be the boundary of E.



(a) Is $\iint_R x\,dS$ positive, negative, or zero?

There is more surface of R at negative values of x. (The plane x+z=4 slices off more of the positive x values.)

(b) Compute $\iint_R \underbrace{\langle x^2, y^2, z^2\rangle}_{\vec{F}} \cdot d\mathbf{S}$.

$$= \iiint_E \operatorname{div}\vec{F}\,dV \quad \text{by Divergence Theorem}$$

$$= \iiint_E (2x+2y+2z)\,dx\,dy\,dz$$

$$= \iint_{x^2+y^2\le 4} \left(\int_0^{4-x} (2x+2y+2z)\,dz \right) dx\,dy \qquad \text{by sticks}$$

$$= \iint_{x^2+y^2\le 4} \left(\underbrace{2x(4-x)}_{8x-2x^2} + \underbrace{2y(4-x)}_{8y-2xy} + \underbrace{(4-x)^2}_{16-8x+x^2} \right) dx\,dy$$

$$= \iint_{x^2+y^2\le 4} \left(-x^2 + \underbrace{8y-2xy}_{\text{integrates to 0 by symmetry}} + \underbrace{16}_{16\times\text{Area(disk)}} \right) dx\,dy$$

$$= \left(\iint_{x^2+y^2\le 4} (-x^2)\,dx\,dy \right) + 0 + 16\cdot 4\pi$$

$$= \left(\int_0^{2\pi}\int_0^2 -r^2\cos^2\theta \cdot r\,dr\,d\theta \right) + 64\pi$$

$$= -4\int_0^{2\pi} \tfrac{1}{2}(1+\cos(2\theta))\,d\theta \quad + 64\pi$$

$$= 60\pi$$

2. A certain vector field $\mathbf{F}$ has a singularity at $(0,0,0)$ (and no others). Other than at the singularity, Mathematica computes that $\operatorname{div}\mathbf{F} = 3$. You center a sphere S of radius 2 at the origin, and Mathematica tells you that

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = 1.$$

But then your computer explodes. You don't remember the formula for $\mathbf{F}$. Your boss demands to know the flow of $\mathbf{F}$ across a sphere of radius 7 (centered at the origin). Can you tell him? (You have 2 minutes.)

Let E be the 3D region between the spheres of radius 2 & 7.

By the divergence theorem,

$$\text{flow out of } E = \iiint_E \operatorname{div}\vec{F}\, dV = \iiint_E 3\, dV = 3\cdot \operatorname{Vol}(E)$$
$$= 3\left(\tfrac{4}{3}\pi(7)^3 - \tfrac{4}{3}\pi(2)^3\right)$$
$$= 1340\pi$$

The flow across the inner sphere is 1 into E,

so the net flow across the outer sphere must

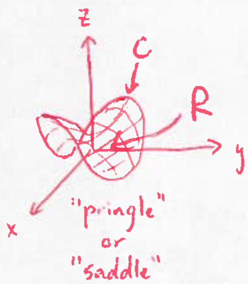
be $\boxed{1340\pi + 1 \text{ out of } E}$.

(out of the sphere of radius 7.)

3. Compute the net flow of $\mathbf{F}(x,y,z) = \langle x+y+z, \sin y, 2x\rangle$ along the curve C that is the intersection of the surfaces $z = x^2 - y^2$ and $x^2+y^2 = 4$.

Too ugly to do directly. Stokes?

$$\text{curl}\,\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ x+y+z & \sin y & 2x \end{vmatrix} = \langle 0-0, -(2-1), 0-1\rangle = \langle 0,-1,-1\rangle.$$



Surface R: $z = x^2-y^2$ with $x^2+y^2 \le 4$.

Parametrize: $x=u \quad y=v \quad z=u^2-v^2$

$$\vec{dS} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2u \\ 0 & 1 & -2v \end{vmatrix} = \langle -2u, 2v, \underset{\sim}{1}\rangle$$

upward normal

$\Rightarrow$ C must be parametrized counterclockwise when viewed from above.

$$\int_C \vec{F}\cdot\vec{dr} = \iint_R \text{curl}\,\vec{F}\cdot\vec{dS}$$

$$= \iint_{u^2+v^2\le 4} \langle 0,-1,-1\rangle\cdot\langle -2u, 2v, 1\rangle\, du\,dv$$

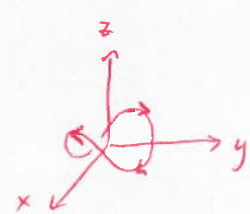
$$= \iint_{u^2+v^2\le 4} (-2v-1)\,du\,dv$$

symmetry

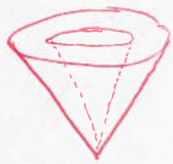
$$= 0 \quad -\text{Area}(u^2+v^2\le 4)$$

$$= -4\pi.$$

So flow along C is 4π clockwise-when-viewed-from-above.



4. Find the volume of the region E that is inside the cone $z = \sqrt{x^2+y^2}$, outside the cone $z = \sqrt{3x^2+3y^2}$, and below $z = 2$.



Spherical:

$z \geq \sqrt{x^2+y^2} \longleftrightarrow \varphi \leq \frac{\pi}{4}$

$z \leq \sqrt{3x^2+3y^2} \longleftrightarrow \varphi \geq \frac{\pi}{6}$

$\left(\rho\cos\varphi \leq \sqrt{3}\cdot\rho\sin\varphi \iff \frac{\sqrt{3}}{3} \leq \tan\varphi \iff \varphi \geq \frac{\pi}{6} \right)$

$z \leq 2 \longleftrightarrow \rho \leq 2\sec\varphi$

$$\text{Vol} = \iiint_E 1 \, dx\,dy\,dz$$

$$= \int_0^{2\pi}\int_{\pi/6}^{\pi/4}\int_0^{2\sec\varphi} \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

$$= 2\pi \cdot \frac{1}{3} \int_{\pi/6}^{\pi/4} 8\sec^3\varphi \sin\varphi \, d\varphi$$

$$= \frac{2\pi}{3} \int_{\pi/6}^{\pi/4} \frac{8\sin\varphi}{\cos^3\varphi} \, d\varphi \qquad u = \cos\varphi, \quad du = -\sin\varphi \, d\varphi$$

$$= \frac{16\pi}{3} \int_{\sqrt{3}/2}^{\sqrt{2}/2} \frac{-du}{u^3}$$

$$= \frac{8\pi}{3} \left[\frac{1}{u^2} \right]_{\sqrt{3}/2}^{\sqrt{2}/2}$$

$$= \frac{8\pi}{3} \left(2 - \frac{4}{3} \right)$$

$$= \frac{16\pi}{9}.$$

5. The 3D region E has a 2D boundary R. The volume of E is 8m^3 and the surface area of R is 7m^2.

(a) Find $\iint_R 3\,dS$.

$= 3 \cdot SA(R) = 21$.

(b) Find $\iiint_E -2\,dV$

$= -2 \cdot Vol(E) = -16$

(c) Find the net flow of $\mathbf{F}(x,y,z) = \langle xe^y, -e^y, 5z + \sin x\rangle$ across R.

$= \iiint_E \operatorname{div} \vec{F}\; dV$

$= \iiint_E \left((e^y) + (-e^y) + (5)\right) dV$

$= \iiint_E 5\, dV = 5 \cdot Vol(E) = 40$

outward.

(By Divergence Thm)

6. Compute $\iint_R x^2\,dS$, where R is the portion of the paraboloid $z = 9x^2 + 4y^2$ with $z \leq 1$.

Parametrize: $x = 2u$, $y = 3v$, $z = 36u^2 + 36v^2$, $36u^2+36v^2 \leq 1$

$$\vec{dS} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & 72u \\ 0 & 3 & 72v \end{vmatrix} = \langle -216u, -144v, 6 \rangle$$

$$dS = \|\vec{dS}\| = \sqrt{(216)^2u^2 + (144)^2v^2 + 36}$$

$$\iint_R x^2\,dS = \iint_{36u^2+36v^2\leq 1} (2u)^2 \underbrace{\sqrt{(216)^2u^2+(144)^2v^2+36}}_{6\sqrt{(36)^2u+(24)^2v^2+1}}\,du\,dv$$

$u = \frac{1}{36} r\cos\theta$ $\quad 0 \leq r \leq 1$

$v = \frac{1}{36} r\sin\theta$ $\quad 0 \leq \theta \leq 2\pi$

$J = \frac{1}{(36)^2} r$

$$= \int_0^{2\pi}\int_0^1 \left(\frac{1}{18^2} r^2\cos^2\theta\right)\left(6\sqrt{r^2\cos^2\theta + \left(\tfrac{2}{3}\right)^2 r^2\sin^2\theta + 1}\right)\left(\frac{1}{36^2}\right) r\,dr\,d\theta$$

Stop here for "set up" as rectangular integral

$$= \int_0^{2\pi}\int_0^1 \frac{6}{(18^2)(36^2)} r^3\cos^2\theta\sqrt{\tfrac{5}{9}r^2\cos^2\theta + \tfrac{13}{9}}\,dr\,d\theta$$

$u = \frac{5}{9} r^2\cos^2\theta + \frac{13}{9}$ (function of r; θ fixed)

$du = \frac{10}{9} r\cos^2\theta\,dr$

$$= \int_0^{2\pi} \frac{6}{18^2\cdot 36^2}\left(\int_{13/9}^{\frac{5}{9}\cos^2\theta+\frac{13}{9}} \frac{9}{10}\left(\tfrac{9}{5}\left(u-\tfrac{13}{9}\right)\sec^2\theta\right)\sqrt{u}\,du\right)d\theta$$

$$= \int_0^{2\pi} \frac{6\cdot 9\cdot 9}{10\cdot 18^2\cdot 36^2\cdot 5}\sec^2\theta \int_{13/9}^{\frac{5}{9}\cos^2\theta+\frac{13}{9}} \left(u^{3/2} - \tfrac{13}{9}u^{1/2}\right)du\,d\theta$$

OK, that's enough of that.

Moving on...

7. Let $\mathbf{F} = \langle m, n, p \rangle$ denote a 3D vector field, and f denote a scalar function of 3 variables. For each of the following, either explain why the expression is meaningless or simplify the expression (using only partial derivatives, f, and m, n, p.

(a) $\nabla \cdot f$ Nonsense: f is not a vector function, so you cannot dot product with it.

(b) $\nabla \cdot \mathbf{F} = \partial_x m + \partial_y n + \partial_z p$ $(= \operatorname{div} \vec{F})$

(c) $\nabla f = \langle \partial_x f, \partial_y f, \partial_z f \rangle$ (= "gradient of f")

(d) $\nabla \mathbf{F}$ Nonsense: vector fields don't <u>have</u> gradients; or, "cannot multiply vectors"

(e) $\nabla \times f$ Nonsense: cannot cross with non-vector f.

(f) $\nabla \times \mathbf{F} = \langle \partial_y p - \partial_z n, -(\partial_x p - \partial_z m), \partial_x n - \partial_y m \rangle$ $(= \operatorname{curl} \vec{F})$

(g) $\nabla \cdot \nabla f = \nabla \cdot \langle \partial_x f, \partial_y f, \partial_z f \rangle = \partial_{xx} f + \partial_{yy} f + \partial_{zz} f$ (= Laplacian of f)

(h) $\nabla \cdot (\nabla \times \mathbf{F}) = \nabla \cdot$ (answer from (f))

$= (\partial_{xy} p - \partial_{xz} n) - (\partial_{yx} p - \partial_{yz} m) + (\partial_{zx} n - \partial_{zy} m) = 0$ if $\vec{F}$ is nice enough $(= \operatorname{div}(\operatorname{curl} \vec{F}))$

(i) $\nabla \times (\nabla \cdot \mathbf{F})$

Nonsense: $\nabla \cdot \vec{F}$ is a scalar function (see (b)), and cannot be crossed with ∇.

(j) $\nabla(\nabla \cdot \mathbf{F}) = \nabla(\partial_x m + \partial_y n + \partial_z p)$

$= \langle \partial_{xx} m + \partial_{xy} n + \partial_{xz} p, \partial_{yx} m + \partial_{yy} n + \partial_{yz} p, \partial_{zx} m + \partial_{zy} n + \partial_{zz} p \rangle$

(k) $\nabla \times (\nabla f)$

$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ \partial_x f & \partial_y f & \partial_z f \end{vmatrix} = \langle \partial_{yz} f - \partial_{zy} f, -(\partial_{xz} f - \partial_{zx} f), \partial_{xy} f - \partial_{yx} f \rangle$

$= \langle 0, 0, 0 \rangle$ if f is nice enough* (= curl (gradient field))

8. Compute the volume contained inside the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. (Hint: use two transformations.)

$u = \frac{x}{a}$, $v = \frac{y}{b}$, $w = \frac{z}{c}$ $\qquad u^2+v^2+w^2 \le 1$

$x = au \quad y = bv \quad z = cw$

$$J = \begin{array}{c} \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{ccc} u & v & w \end{array} \\ \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} \end{array} = abc$$

$$\text{Vol} = \iiint_{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \le 1} 1 \, dx\,dy\,dz$$

$$= \iiint_{u^2+v^2+w^2 \le 1} abc \, du\,dv\,dw$$

spherical u, v, w in ρ, φ, θ

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{1} abc\, \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

$$= \cdots = \boxed{\frac{4abc\pi}{3}}$$

– OR –

$= abc \cdot \text{Vol}$ (inside unit sphere)

$= abc \cdot \frac{4}{3}\pi (1)^3$

9. Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be constant vectors, $\mathbf{r} = \langle x, y, z \rangle$, and let E be the region defined by

$$0 \le \mathbf{a}\cdot\mathbf{r} \le \alpha, \qquad 0 \le \mathbf{b}\cdot\mathbf{r} \le \beta, \qquad 0 \le \mathbf{c}\cdot\mathbf{r} \le \gamma.$$

Prove that

$$\iiint_E (\mathbf{a}\cdot\mathbf{r})(\mathbf{b}\cdot\mathbf{r})(\mathbf{c}\cdot\mathbf{r})\,dx\,dy\,dz = \frac{(\alpha\beta\gamma)^2}{8\,|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|}.$$

(Hint: we didn't mention it but your notebooks did: the volume of the parallelepiped spanned by $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is given by $|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|$, which is also the absolute value of determinant of the matrix whose rows are $\mathbf{a}, \mathbf{b}, \mathbf{c}$.)

Let $u = a\cdot r = a_1 x + a_2 y + a_3 z$ so $0 \le u \le \alpha$

$v = b \cdot r$ $\qquad 0 \le v \le \beta$

$w = c \cdot r$ $\qquad 0 \le w \le \gamma$

$$\frac{1}{J} = \begin{array}{c|ccc|} & x & y & z \\ u & a_1 & a_2 & a_3 \\ v & \vdots & & \\ w & \vdots & \ddots & \end{array} = a\cdot(b\times c) \quad \text{from hint.}$$

$$\iiint_E (a\cdot r)(b\cdot r)(c\cdot r)\,dx\,dy\,dz$$

$$= \int_0^{\gamma}\int_0^{\beta}\int_0^{\alpha} uvw \cdot \left|\frac{1}{a\cdot(b\times c)}\right| du\,dv\,dw$$

$$= \frac{1}{|a\cdot(b\times c)|} \int_0^{\alpha} u\,du \cdot \int_0^{\beta} v\,dv \cdot \int_0^{\gamma} w\,dw$$

$$= \frac{1}{|a\cdot(b\times c)|} \cdot \frac{1}{8} \cdot \alpha^2\beta^2\gamma^2 .$$

10. Let $\mathbf{F}(x,y,z) = \langle \overbrace{e^y + yz\cos(x) + y\cos(xy) + yz^2}^{m}, \overbrace{xe^y + z\sin(x) + x\cos(xy) + xz^2 + e^z}^{n}, \overbrace{y\sin(x) + 2xyz + ye^z}^{p} \rangle$.

(a) Use the Gradient Test to verify that $\mathbf{F}$ is a gradient field.

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ m & n & p \end{vmatrix} = \Big\langle (\sin x + 2xz + e^z) - (0 + \sin x + 0 + 2xz + e^z),$$

$$-(y\cos x + 2yz + 0) + (0 + y\cos x + 0 + 2yz),$$

$$(e^y + z\cos x + (\cos(xy) - xy\sin(xy)) + z^2 + 0) - (e^y + z\cos x + (\cos(xy) - xys\ldots + z^2) \Big\rangle$$

$$= \langle 0,0,0 \rangle \checkmark$$

(b) Find a potential function for $\mathbf{F}$.

$F = \nabla f \Rightarrow m = \partial_x f, \quad n = \partial_y f, \quad p = \partial_z f$

$\Downarrow$

$f = xe^y + yz\sin x + \sin(xy) + xyz^2 + g(y,z)$ "constant of integration with respect to x"

$\Rightarrow \partial_y f = xe^y + z\sin x + x\cos(xy) + xz^2 + \partial_y g = n$

$\Rightarrow \partial_y g = e^z \Rightarrow g = ye^z + h(z)$

$\Rightarrow f = xe^y + yz\sin x + \sin(xy) + xyz^2 + ye^z + h(z)$

$\Rightarrow \partial_z f = y\sin x + 2xyz + ye^z + h'(z) = p$

$\Rightarrow h'(z) = 0 \Rightarrow h = \text{constant}$, take $h = 0$

$f = xe^y + yz\sin x + \sin(xy) + xyz^2 + ye^z.$

(c) Compute the flow of $\mathbf{F}$ along the curve with parametrization $\ell(t) = \langle \pi t, t^2 - t, t^3 \rangle$, $t \in [0,1]$.

By the Fundamental Theorem of Path Integrals, start $= \ell(0) = \langle 0,0,0 \rangle$, end $= \ell(1) = \langle \pi, 0, 1 \rangle$

$$\text{flow along } C = \int_C F \cdot d\vec{r} = f(\text{end}) - f(\text{start})$$

$$= f(\pi,0,1) - f(0,0,0)$$

$$= (\pi + 0 + 0 + 0 + 0) - (0 + 0 + 0 + 0 + 0)$$

$$= \pi.$$