Name: ____________________

- **READ THE FOLLOWING DIRECTIONS!**
- **Do NOT open the exam until instructed to do so.**
- You have until 12:50 to complete this exam. When you are told to stop writing, do it or you will lose all points on the page you write on.
- You may not communicate with other students during this test.
- No written materials of any kind are allowed. No scratch paper is allowed except as given by the proctors.
- No phones, calculators, or any other electronic devices are allowed for any reason, including checking the time (a simple wristwatch is fine).
- Any case of cheating will be taken extremely seriously.
- Show all your work and explain your answers.
- Before turning in your exam, check to make certain you've answered all the questions.
- You do not need to simplify algebraic expressions.

Some possibly useful formulas:

$$\cos^2 t = \frac{1}{2}(1 + \cos(2t))$$
$$\sin^2 t = \frac{1}{2}(1 - \cos(2t))$$

1. Let E be the 3D region bounded by the cylinder $x^2 + y^2 = 4$ and the planes $z = 0$ and $x + z = 4$. Let R be the boundary of E.

   (a) Is $\iint_R x\, dS$ positive, negative, or zero?

   (b) Compute $\iint_R \langle x^2, y^2, z^2 \rangle \cdot \mathbf{dS}$.

2. A certain vector field $\mathbf{F}$ has a singularity at $(0, 0, 0)$ (and no others). Other than at the singularity, Mathematica computes that $\operatorname{div} \mathbf{F} = 3$. You center a sphere S of radius 2 at the origin, and Mathematica tells you that

$$\iint_S \mathbf{F} \cdot \mathbf{dS} = 1.$$

But then your computer explodes. You don't remember the formula for $\mathbf{F}$. Your boss demands to know the flow of $\mathbf{F}$ across a sphere of radius 7 centered at the origin. Can you tell him? (You have 2 minutes.)

3. Compute the net flow of $\mathbf{F}(x, y, z) = \langle x + y + z, \sin y, 2x \rangle$ along the curve C that is the intersection of the surfaces $z = x^2 - y^2$ and $x^2 + y^2 = 4$.

4. Find the volume of the region that is inside the cone $z = \sqrt{x^2 + y^2}$, outside the cone $z = \sqrt{3x^2 + 3y^2}$, and below $z = 2$.

5. The solid 3D region E has a boundary surface R. The volume of E is 8m^3, and the surface area of R is 7m^2.

   (a) Find $\iint_R 3\,dS$.

   (b) Find $\iiint_E -2\,dV$

   (c) Find the net flow of $\mathbf{F}(x, y, z) = \langle xe^y, -e^y, 5z + \sin x\rangle$ across R.

6. Rewrite $\iint_R x^2\,dS$ as an integral over some rectangle in a 2-dimensional space, where R is the portion of the paraboloid $z = 9x^2 + 4y^2$ with $z \leq 1$. (Start to compute the desired integral; it will get ugly, so stop whenever you've reduced it to a double integral over a rectangle.)

7. Let $\mathbf{F} = \langle m, n, p \rangle$ denote a 3D vector field, and f denote a scalar function of 3 variables. For each of the following, either explain why the expression is meaningless or simplify the expression (using only partial derivatives, f, and m, n, p.

   (a) $\nabla \cdot f$

   (b) $\nabla \cdot \mathbf{F}$

   (c) ∇f

   (d) $\nabla \mathbf{F}$

   (e) $\nabla \times f$

   (f) $\nabla \times \mathbf{F}$

   (g) $\nabla \cdot \nabla f$

   (h) $\nabla \cdot (\nabla \times \mathbf{F})$

   (i) $\nabla \times (\nabla \cdot \mathbf{F})$

   (j) $\nabla(\nabla \cdot \mathbf{F})$

   (k) $\nabla \times (\nabla f)$

8. Compute the volume contained inside the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$. (Hint: use two transformations.)

9. Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be constant vectors, $\mathbf{r} = \langle x, y, z \rangle$, and let E be the region defined by

$$0 \leq \mathbf{a} \cdot \mathbf{r} \leq \alpha, \qquad 0 \leq \mathbf{b} \cdot \mathbf{r} \leq \beta, \qquad 0 \leq \mathbf{c} \cdot \mathbf{r} \leq \gamma.$$

Prove that

$$\iiint_E (\mathbf{a} \cdot \mathbf{r})(\mathbf{b} \cdot \mathbf{r})(\mathbf{c} \cdot \mathbf{r})\, dx\, dy\, dz = \frac{(\alpha\beta\gamma)^2}{8\,|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|}.$$

(Hint: we didn't mention it but your notebooks did: the volume of the parallelepiped spanned by $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is given by $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$, which is also the absolute value of the determinant of the matrix whose rows are $\mathbf{a}, \mathbf{b}, \mathbf{c}$.)

10. Let $\mathbf{F}(x,y,z) = \langle e^y + yz\cos(x) + y\cos(xy) + yz^2, xe^y + z\sin(x) + x\cos(xy) + xz^2 + e^z, y\sin(x) + 2xyz + ye^z \rangle$.

(a) Use the Gradient Test to verify that $\mathbf{F}$ is a gradient field.

(b) Find a potential function for $\mathbf{F}$.

(c) Compute the flow of $\mathbf{F}$ along the curve with parametrization $\ell(t) = \langle \pi t, t^2 - t, t^3 \rangle$, $t \in [0,1]$.