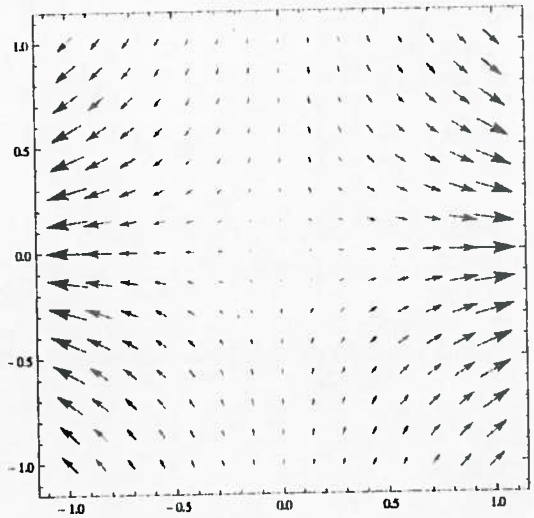
Name: Solutions

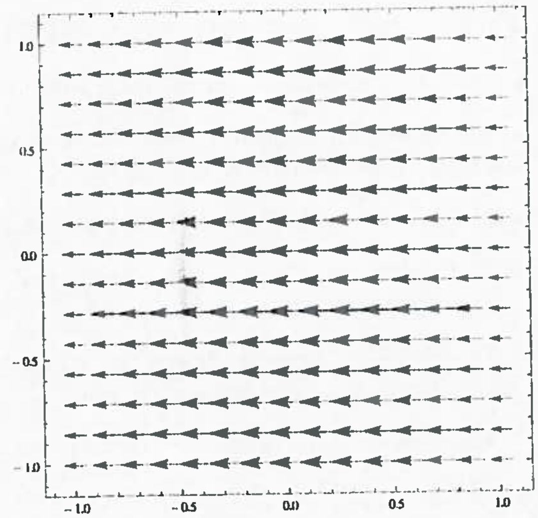
- **READ THE FOLLOWING DIRECTIONS!**
- **Do NOT open the exam until instructed to do so.**
- You have until 12:50 to complete this exam. When you are told to stop writing, do it or you will lose all points on the page you write on.
- You may not communicate with other students during this test.
- No written materials of any kind are allowed. No scratch paper is allowed except as given by the proctors.
- No phones, calculators, or any other electronic devices are allowed for any reason, including checking the time (a simple wristwatch is fine).
- Any case of cheating will be taken extremely seriously.
- Show all your work and explain your answers.
- Before turning in your exam, check to make certain you've answered all the questions.
- You do not need to simplify algebraic expressions.

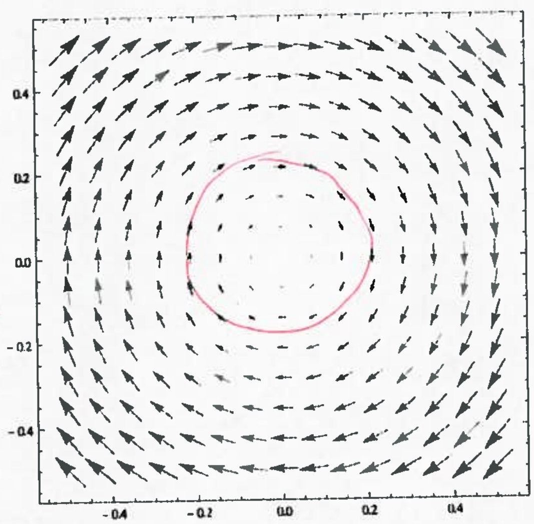
Some possibly useful formulas:

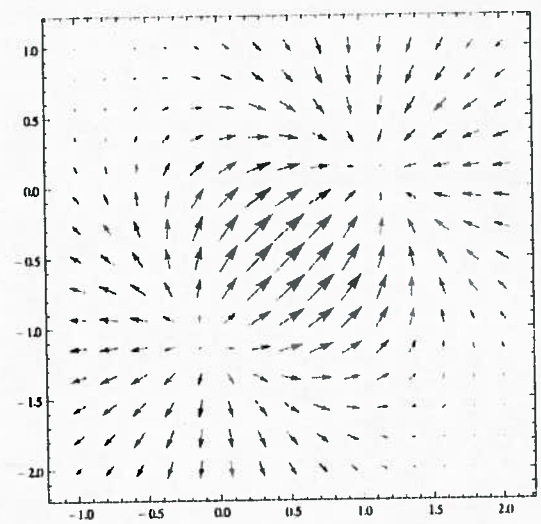
$$\cos^2 t = \frac{1}{2}(1 + \cos(2t))$$
$$\sin^2 t = \frac{1}{2}(1 - \cos(2t))$$

1. All but one of the following vector fields are gradient fields. Which one can't be a gradient field? **Why?**









This one cannot be conservative: the flow along the indicated closed curve is nonzero (clockwise).

2. Let $\mathbf{F}(x,y) = \langle ye^x + e^x + x^2 - 1, e^x + y^4 + \sin(\pi y)\rangle$.

(a) Find a potential function for $\mathbf{F}$.

If $\vec{F} = \nabla f$,

$\partial_x f = ye^x + e^x + x^2 - 1$ and $\partial_y f = e^x + y^4 + \sin(\pi y)$

$\Rightarrow f = ye^x + e^x + \frac{1}{3}x^3 - x + g(y)$

$\Rightarrow \partial_y f = e^x + 0 + 0 - 0 + g'(y) = e^x + y^4 + \sin(\pi y)$

$\Rightarrow g'(y) = y^4 + \sin(\pi y)$

$\Rightarrow g(y) = \frac{1}{5}y^5 - \frac{1}{\pi}\cos(\pi y) + C$

$f(x,y) = ye^x + e^x + \frac{1}{3}x^3 - x + \frac{1}{5}y^5 - \frac{1}{\pi}\cos(\pi y)$ is a potential function.

(b) Compute the flow of $\mathbf{F}$ along the part of the curve $y = \cos(\pi x)$ going from $(0,1)$ to $(3,-1)$.

Since $\vec{F}$ is conservative (no singularities!),

$$\int_C \vec{F} \cdot \langle dx, dy\rangle = f(3,-1) - f(0,1)$$

$$= -e^3 + e^3 + \frac{1}{3}3^3 - 3 + \frac{1}{5}(-1)^5 - \frac{1}{\pi}\cos(-\pi)$$

$$- \left(1 + 1 + 0 - 0 + \frac{1}{5} - \frac{1}{\pi}\cos(\pi)\right)$$

$$= 6 - \frac{1}{5} + \frac{1}{\pi} - \left(2 + \frac{1}{5} + \frac{1}{\pi}\right)$$

$$= 4 - \frac{2}{5}$$

$$\left(= \frac{18}{5}\right)$$

3. Find all sources and sinks of the vector field $\mathbf{F}(x, y) = \langle y^4, xy^3e^x \rangle$.

$$\operatorname{div}\vec{F} = \nabla \cdot \vec{F} = \partial_x(y^4) + \partial_y(xy^3e^x)$$

$$= 0 + 3xy^2e^x.$$

$3xy^2e^x = 0 \iff x=0$ -or- $y=0$

$> 0 \iff x>0$, & $y \neq 0$

$< 0 \iff x<0$, & $y \neq 0$

(no singularities to check)

4. Find all sources and sinks of the vector field $\mathbf{G}(x, y) = \left\langle \frac{x+y}{x^2+y^2}, \frac{y-x}{x^2+y^2} \right\rangle$.

$$\operatorname{div} \vec{G} = \nabla \cdot \vec{G} = \partial_x \left(\frac{x+y}{x^2+y^2} \right) + \partial_y \left(\frac{y-x}{x^2+y^2} \right)$$

$$= \frac{(x^2+y^2)(1) - (x+y)(2x)}{(x^2+y^2)^2} + \frac{(x^2+y^2)(1) - (y-x)(2y)}{(x^2+y^2)^2}$$

$$= 0 \quad (\text{except @ } (0,0)).$$

So no points except perhaps the singularity at $(0,0)$ are sources or sinks.

The flow across the circle $C(r)$ of radius r centered at $(0,0)$ is

$$\int_{C(r)} \vec{G} \cdot \langle dy, -dx \rangle$$

$$= \int_0^{2\pi} \left\langle \frac{r\cos t + r\sin t}{r^2}, \frac{r\sin t - r\cos t}{r^2} \right\rangle \cdot \langle r\cos t, r\sin t \rangle \, dt$$

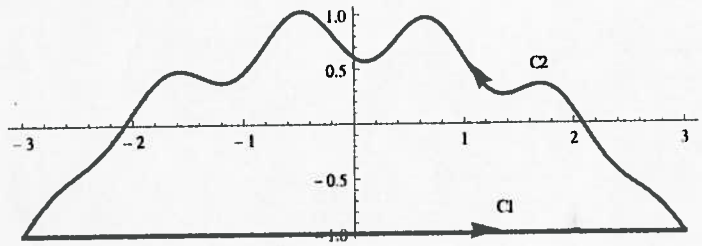
$$= \int_0^{2\pi} \left(\cos^2 t + \sin t \cos t + \sin^2 t - \sin t \cos t \right) dt$$

$$= \int_0^{2\pi} 1 \, dt$$

$$= 2\pi.$$

So $(0,0)$ is a source.

5. Let $\mathbf{F}(x,y) = \langle e^x y^2, 2e^x y - 1\rangle$. The curve C consists of two parts, C_1 and C_2, as shown below.



(a) Find the flow of $\mathbf{F}$ along C.

$$\operatorname{rot}\vec{F} = \begin{vmatrix} \partial_x & \partial_y \\ e^x y^2 & 2e^x y - 1 \end{vmatrix} = \partial_x(2e^x y - 1) - \partial_y(e^x y^2)$$

$$= 2e^x y - 2e^x y = 0.$$

$$\text{flow-along} = \iint_{\text{inside } C} \operatorname{rot}\vec{F}\, dA = \iint 0\, dA = 0.$$

(no singularities)

(b) Find the flow of $\mathbf{F}$ along C_1.

Parametrize C_1: $x = t$, $y = -1$, $t \in [-3,3]$.

$$\text{flow-along} = \int_{C_1} \vec{F}\cdot\langle dx, dy\rangle = \int_{-3}^{3} \langle e^t(-1)^2, 2e^t(-1)-1\rangle \cdot \langle 1, 0\rangle\, dt$$

$$= \int_{-3}^{3} e^t\, dt = e^3 - e^{-3}.$$ (>0, flow is left-to-right)

(c) Find the flow of $\mathbf{F}$ along C_2.

$$\int_C F\cdot\langle dx,dy\rangle = \int_{C_1} F\cdot\langle dx,dy\rangle + \int_{C_2} F\cdot\langle dx,dy\rangle$$

$$\Rightarrow \int_{C_2} F\cdot\langle dx,dy\rangle = \int_C F\cdot\langle dx,dy\rangle - \int_{C_1} F\cdot\langle dx,dy\rangle$$

$$= \underset{(a)}{0} - \underset{(b)}{(e^3 - e^{-3})}$$

$$= e^{-3} - e^3.$$ (<0, flow is left-to-right)

6. Let $\mathbf{F}(x, y) = \langle x + y, \frac{1}{2}y^2\rangle$. Let C be the curve going from $(1, 0)$ to $(0, 2)$ along the parabola $4 - 4x = y^2$, then from $(0, 2)$ to $(-1, 0)$ along the parabola $4 + 4x = y^2$, then from $(-1, 0)$ to $(1, 0)$ along the x-axis. Let R be the region bounded by C.

(a) Explain why you can measure the flow of $\mathbf{F}$ across C by the double integral

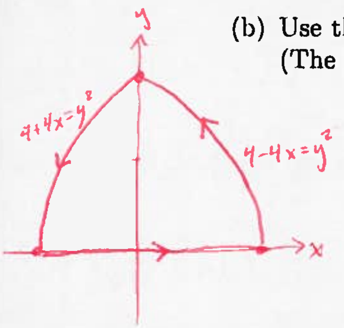
$$\iint_R (1 + y)\, dx\, dy.$$

Gauss-Green: $\vec{F}$ has no singularities, so

$$\text{flow of } \vec{F} \text{ across } C = \iint_R \operatorname{div}\vec{F}\, dx dy$$

$$= \iint_R \left(\partial_x(x+y) + \partial_y\left(\tfrac{1}{2}y^2\right)\right) dx dy = \iint_R (1+y)\, dx dy.$$

(b) Use the transformation $x = u^2 - v^2$ and $y = 2uv$ (assume $u, v \geq 0$) to compute the above integral. (The region is a little tricky.) Which direction is the flow of $\mathbf{F}$ across C?



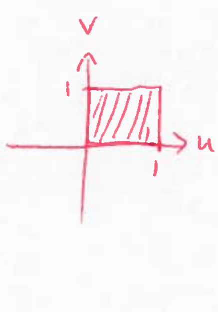
(x,y) (u,v)

$$J = \begin{array}{c|cc} & \partial u & \partial v \\ \hline x & 2u & -2v \\ y & 2v & 2u \end{array} = 4u^2 + 4v^2 \geq 0$$

$$4 - 4x = y^2 \iff 4 - 4u^2 + 4v^2 = 4u^2v^2$$
$$\iff 1 - u^2 = (u^2 - 1)v^2$$
$$\iff v^2 = \frac{1-u^2}{u^2-1} = -1 \ ↯$$
$$\text{or } u^2 - 1 = 0 \iff u = \pm 1$$
$$u \geq 0 \Rightarrow u = 1$$

$$4 + 4x = y^2 \iff 4 + 4u^2 - 4v^2 = 4u^2v^2$$
$$\iff 1 + u^2 = (u^2+1)v^2$$
$$\iff v^2 = \frac{1+u^2}{u^2+1} = 1 \Rightarrow v = 1 \ (v \geq 0)$$
$$(\text{or } u^2 + 1 = 0 \ ↯)$$

$$y = 0 \iff u = 0 \text{ or } v = 0$$



$$\therefore \iint_R (1+y)\, dx dy = \int_0^1\int_0^1 (1+2uv)(4u^2+4v^2)\, du dv$$

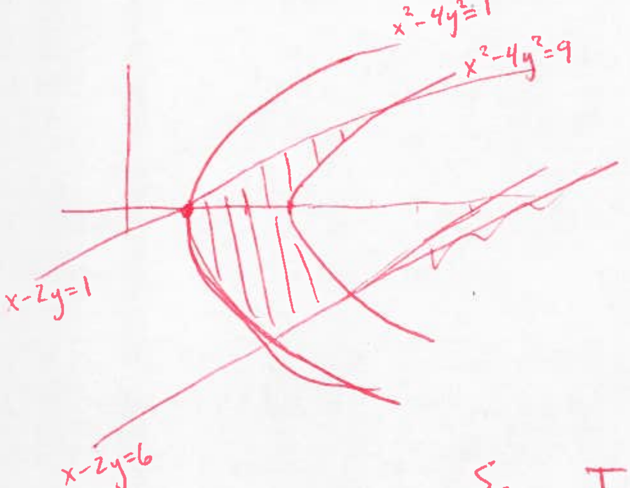
$$= 4\int_0^1\int_0^1 (u^2 + v^2 + 2u^3v + 2uv^3)\, du dv$$

$$= 4\int_0^1 \left(\tfrac{1}{3} + v^2 + \tfrac{1}{2}v + v^3\right) dv$$

$$= 4\left(\tfrac{1}{3} + \tfrac{1}{3} + 1 + \tfrac{1}{4}\right)$$

$$\left(= \tfrac{23}{3}\right).$$

7. Let R be the region ~~in the first quadrant~~ bounded by the curves $x^2 - 4y^2 = 1$, $x^2 - 4y^2 = 9$, $x - 2y = 1$, and $x - 2y = 6$. Compute the area of R.



Let $u = x^2-4y^2 \qquad 1 \le u \le 9$

$v = x-2y. \qquad 1 \le v \le 6$

Then $u = x^2-4y^2 = (x-2y)(x+2y) = v(x+2y)$,

i.e. $\left.\begin{array}{l} x+2y = \frac{u}{v} \\ x-2y = v \end{array}\right\} \Rightarrow x = \frac{1}{2}\left(\frac{u}{v}+v\right)$

$\Rightarrow y = \frac{1}{4}\left(\frac{u}{v}-v\right)$.

So
$$J = \begin{array}{c|cc} & \partial u & \partial v \\ \hline x & \frac{1}{2v} & \frac{1}{2}\left(-\frac{u}{v^2}+1\right) \\ y & \frac{1}{4v} & \frac{1}{4}\left(-\frac{u}{v^2}-1\right) \end{array} = \frac{1}{8}\left(\frac{1}{v}\left(-\frac{u}{v^2}-1\right) - \frac{1}{v}\left(-\frac{u}{v^2}+1\right)\right)$$

$$= \frac{1}{8v}\left(-\frac{u}{v^2} - 1 + \frac{u}{v^2} - 1\right)$$

$$= -\frac{1}{4v}. \qquad v > 0 \Rightarrow |J| = \frac{1}{4v}$$

So
$$\text{Area}(R) = \iint_R 1 \, dx\, dy$$

$$= \int_1^6 \int_1^9 1 \cdot \frac{1}{4v} \, du\, dv$$

$$= \int_1^6 \frac{2}{v} \, dv$$

$$= 2(\ln 6 - \ln 1)$$

$$= 2 \ln 6.$$

8. Compute $\iint_D (x+y)^2\,dx\,dy$ where D is the unit disk.

$$= \iint_D (x^2+y^2+2xy)\,dxdy$$

$$= \int_0^{2\pi}\int_0^1 (r^2 + 2r^2\cos\theta\sin\theta)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{4} + \frac{1}{2}\cos\theta\sin\theta\right) d\theta \qquad \begin{array}{l} u=\sin\theta \\ du=\cos\theta\,d\theta \end{array}$$

$$= \frac{2\pi}{4} + \frac{1}{2}\int_0^0 u\,du$$

$$= \frac{\pi}{2}.$$

$$\text{Area} = \iint_{\text{region}} 1\, dx\, dy$$

9. Compute the area of the interior of the ellipse $4x^2 + 9y^2 = 1$ by two methods: Gauss-Green integration and 2D transformations.

Gauss-Green: $m=0,\ n=x$ $\qquad C: \begin{array}{l} x=\frac{1}{2}\cos t \\ y=\frac{1}{3}\sin t \end{array} \quad t\in[0,2\pi]$

$$\int_C m\,dx + n\,dy = \int_0^{2\pi} \underbrace{\left(\tfrac{1}{2}\cos t\right)}_{n} \underbrace{\left(\tfrac{1}{3}\cos t\right)dt}_{dy}$$

$$= \frac{1}{6}\int_0^{2\pi} \frac{1}{2}(1+\cos(2t))\,dt$$

$$= \frac{1}{12}(2\pi) + 0$$

$$= \frac{\pi}{6}.$$

2D transforms:

$$\iint_{\substack{\text{inside}\\ \text{ellipse}}} 1\, dx\,dy \qquad \begin{array}{l} u=2x \\ v=3y \end{array} \qquad 4x^2+9y^2 \le 1 \iff u^2+v^2\le 1$$

$$J = \begin{vmatrix} 1/2 & 0 \\ 0 & 1/3 \end{vmatrix} = \frac{1}{6}$$

$$= \iint_{\substack{\text{unit}\\ \text{disk}}} 1\cdot\frac{1}{6}\,du\,dv$$

polar: $\begin{array}{l} u=r\cos\theta \\ v=r\sin\theta \end{array} \quad J_2 = r$

$$= \int_0^{2\pi}\int_0^1 \frac{1}{6} r\,dr\,d\theta$$

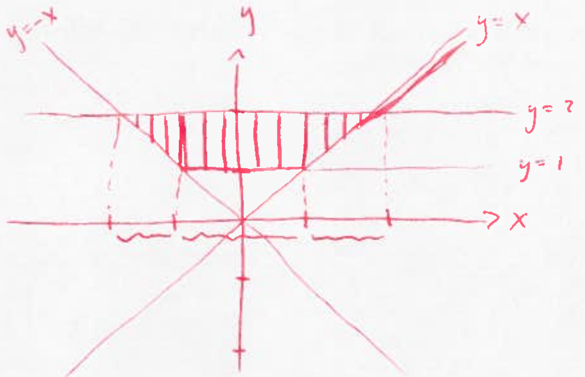
$$= \int_0^{2\pi} \frac{1}{12}\,d\theta$$

$$= \frac{2\pi}{12}$$

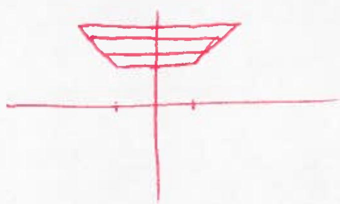
$$= \frac{\pi}{6}.$$

10. Rewrite the following as one double integral by changing the order of integration (/direction of slices).

$$\int_{-2}^{-1}\int_{-x}^{2} f(x,y)\,dx\,dy + \int_{-1}^{1}\int_{1}^{2} f(x,y)\,dx\,dy + \int_{1}^{2}\int_{x}^{2} f(x,y)\,dx\,dy$$



$$\boxed{\int_1^2 \int_{-y}^{y} f(x,y)\, dx\, dy}$$



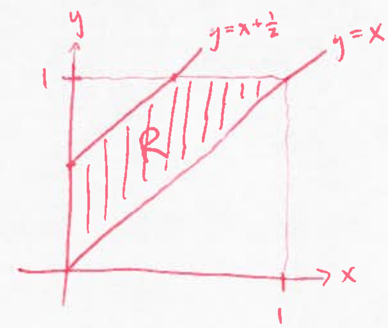
11. Alice and Bob want to meet at the coffee shop to study for their upcoming exam. They agree that "about 2pm" is a good time. For these two, the probability distribution of their arrival time is given by $p(x,y) = 4(1-x)(1-y)$, where x measures the number of hours after 2pm that Alice arrives, and y measures the number of hours after 2pm that Bob arrives, both between 0 and 1 hour. (The distribution is zero outside of this range.)

(1) Bob is impatient, and will leave if Alice isn't already there when he arrives. If Alice (2) arrives and Bob isn't already there, she will wait for up to half an hour, after which she will leave.

What is the probability that the two actually meet to study together?

From (1), a meeting requires $x \le y$

(2) - - - - - - - $y \le x + \frac{1}{2}$



Probability $= \iint_R p(x,y)\,dx\,dy$.

Method 1: straightforward

$$= \int_0^{1/2}\int_x^{x+\frac{1}{2}} 4(1-x)(1-y)\,dy\,dx + \int_{1/2}^1\int_x^1 4(1-x)(1-y)\,dy\,dx$$

$$= \int_0^{1/2} 4(1-x)\left[y-\tfrac{1}{2}y^2\right]_x^{x+\frac{1}{2}}dx + \int_{1/2}^1 4(1-x)\left[y-\tfrac{1}{2}y^2\right]_x^1 dx$$

$$= \int_0^{1/2}\begin{pmatrix}4(1-x)(x+\frac{1}{2}) - 2(1-x)(x+\frac{1}{2})^2 \\ -4(1-x)x + 2(1-x)x^2\end{pmatrix}dx + \int_{1/2}^1\begin{pmatrix}4(1-x) - 2(1-x) \\ -4(1-x)x + 2(1-x)x^2\end{pmatrix}dx$$

$$= \int_0^{1/2}\left(\tfrac{3}{2} - \tfrac{7}{2}x + 2x^2\right)dx + \int_{1/2}^1 2(1-x)^3 dx$$

$$= \frac{3}{4} - \frac{7}{16} + \frac{1}{12} + \frac{1}{32}$$

$$\left(= \frac{10}{12} + \frac{11}{32} = \frac{41}{96}\right).$$

Method 2: Transform

$u = y - x$ ⟹ $y = u + v$

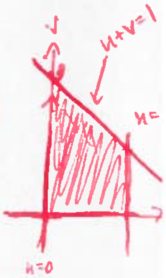
$v = x$ ⟹ $x = v$

(there are other good choices)

$$J = \begin{array}{c|cc} & \partial_u & \partial_v \\ \hline x & 0 & 1 \\ y & 1 & 1 \end{array} = -1$$

$y = x + \frac{1}{2} \longleftrightarrow u = \frac{1}{2}$ $\qquad$ $x = 0 \longleftrightarrow v = 0$

$y = x \longleftrightarrow u = 0$ $\qquad$ $y = 1 \longleftrightarrow u + v = 1$



$$\int_0^{1/2}\int_0^{1-u} 4(1-v)(1-u-v)\cdot 1\,dv\,du$$

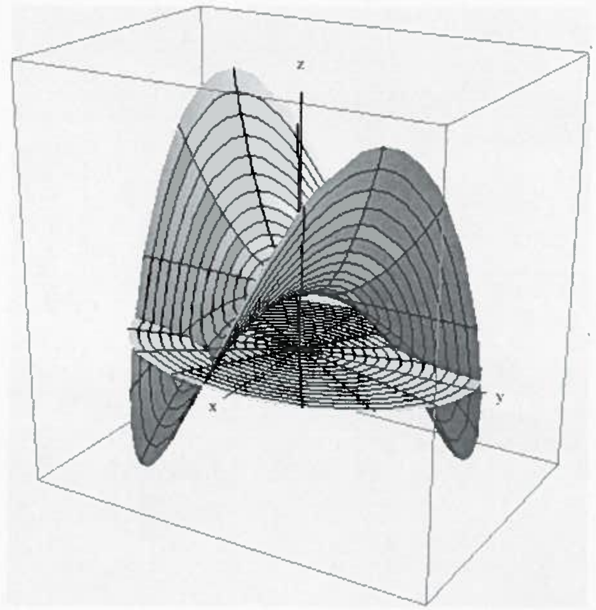
$$= \int_0^{1/2}\int_0^{1-u} 4(1-u-2v+uv+v^2)\,dv\,du$$

$$= \int_0^{1/2} 4\Big(\underbrace{1-u-u+u^2-(1-u)^2}_{0} + \tfrac{1}{2}u(1-u)^2 + \tfrac{1}{3}(1-u)^3\Big)du$$

$$= 4\left(+\frac{1}{6} + \frac{1}{24} + \frac{1}{8\cdot 16} + \frac{1}{12\cdot 16} + \frac{1}{12\cdot 1}\right)$$

$(= 41/96)$.

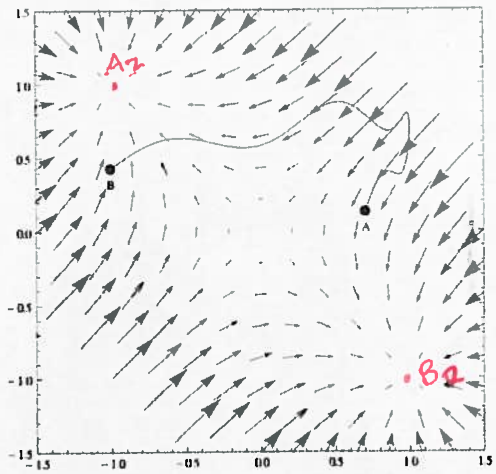
12. Below is shown the graph of a function $f(x, y)$ over a region R in the xy-plane. Without any computation, is $\iint_R f(x, y)\, dA$ positive, negative, or (nearly) zero? Why?



Positive: there is much more space below the surface & above the region R (Positive contribution) than above the surface & below R. (negative contribution)

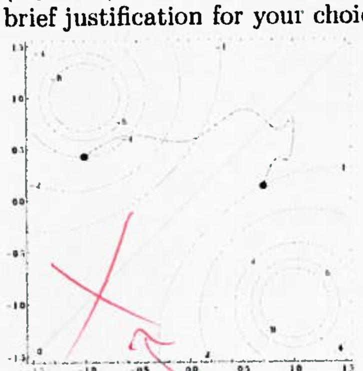
13. Below is shown part of a gradient field **F**, together with a curve C.

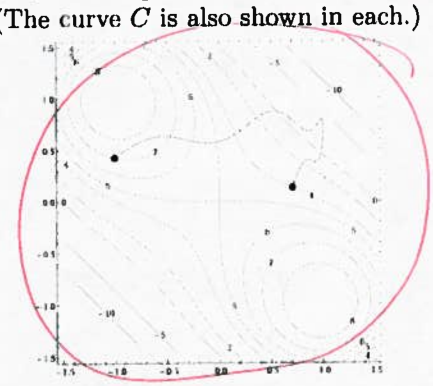
Say $\vec{F} = \nabla f$.

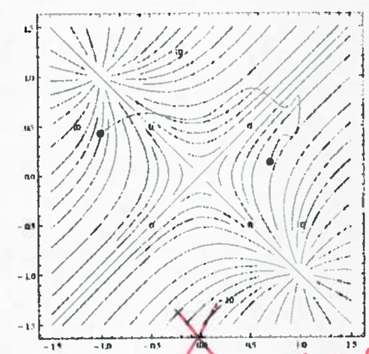


A and B are both local maxima for f.

(a) **(8 points)** One of the following is the contour plot of a potential function for **F**. Circle it. Give a brief justification for your choice. (The curve C is also shown in each.)



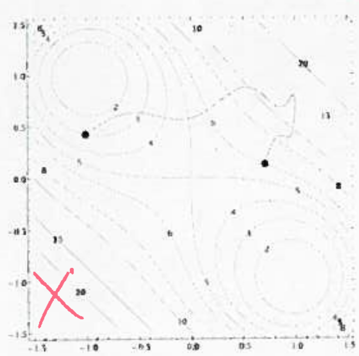


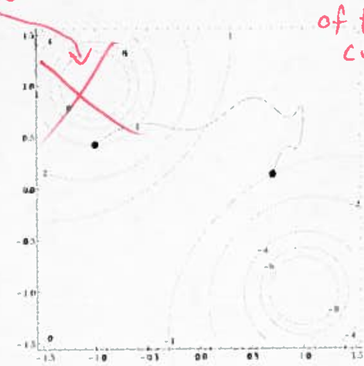


A_2 & B_2 are not maxi… he…

the vectors of $\vec{F}$ are not $\perp$ to all of these curves

A_2 & B_2 are local minima here





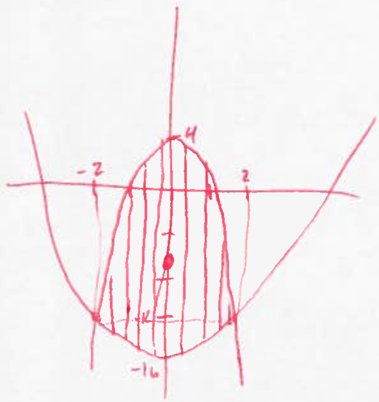
(b) **(10 points)** Find $\int_C \mathbf{F} \cdot \langle dx, dy \rangle$, assuming C is parametrized to go from A to B.

$= f(B) - f(A)$

$= 7 - 4$ (from plot in (a))

$= \boxed{3}$.

14. Find the centroid of the region bounded by the curves $y = x^2 - 16$ and $y = 4 - x^4$.



$$x^2 - 16 = 4 - x^4$$

$$\iff x^4 + x^2 - 20 = 0$$

$$\iff (x^2+5)(x^2-4) = 0$$

$$\iff x^2 = -5 \text{ or } x^2 = 4$$

$$\Rightarrow x = \pm 2$$

$$\Rightarrow y = -12$$

Clearly $\bar{x} = 0$.

$$\text{Area} = \int_{-2}^{2}\int_{x^2-16}^{4-x^4} 1 \, dy\, dx = \int_{-2}^{2} (4 - x^4 - x^2 + 16)\, dx = \left[20x - \tfrac{1}{5}x^5 - \tfrac{1}{3}x^3\right]_{-2}^{2}$$

$$= 80 - \frac{64}{5} - \frac{16}{3}. \quad \left(= \frac{928}{15}\right)$$

$$\bar{y} = \frac{1}{\text{Area}} \int_{-2}^{2}\int_{x^2-16}^{4-x^4} y\, dy\, dx = \frac{1}{2A}\int_{-2}^{2} \left((4-x^4)^2 - (x^2-16)^2\right) dx$$

$$= \frac{1}{2A}\int_{-2}^{2} \left(16 - 8x^4 + x^8 - x^4 + 32x^2 - 256\right) dx$$

$$= \frac{1}{2A}\int_{-2}^{2} \left(-240 + 32x^2 - 9x^4 + x^8\right) dx$$

$$= \frac{1}{2A}\left[-240x + \frac{32}{3}x^3 - \frac{9}{5}x^5 + \frac{1}{9}x^9\right]_{-2}^{2}$$

$$= \frac{15}{1856}\left(-960 + \frac{512}{3} - \frac{9 \cdot 64}{5} + \frac{1024}{9}\right)$$

$$\left(= \frac{15}{1856} \cdot \frac{-35584}{45} = -\frac{556}{87} \approx -6.4\right)$$