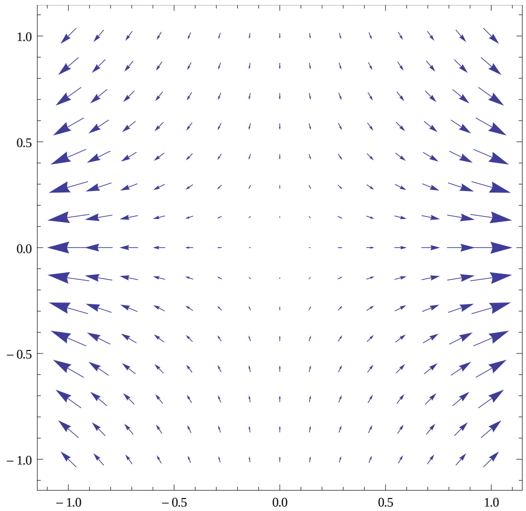
Name: ____

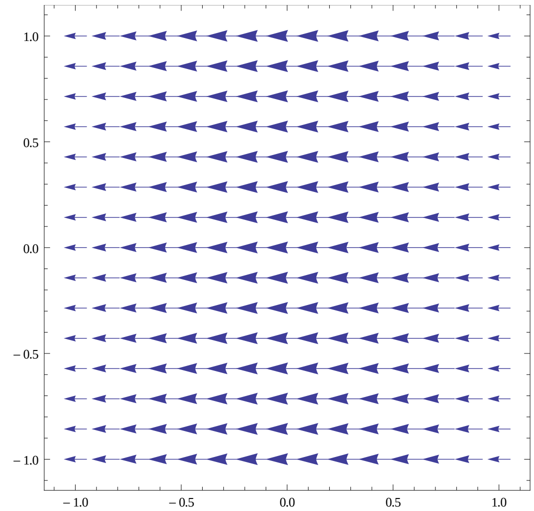
- **READ THE FOLLOWING DIRECTIONS!**
- **Do NOT open the exam until instructed to do so.**
- You have until 12:50 to complete this exam. When you are told to stop writing, do it or you will lose all points on the page you write on.
- You may not communicate with other students during this test.
- No written materials of any kind are allowed. No scratch paper is allowed except as given by the proctors.
- No phones, calculators, or any other electronic devices are allowed for any reason, including checking the time (a simple wristwatch is fine).
- Any case of cheating will be taken extremely seriously.
- Show all your work and explain your answers.
- Before turning in your exam, check to make certain you've answered all the questions.
- You do not need to simplify algebraic expressions.

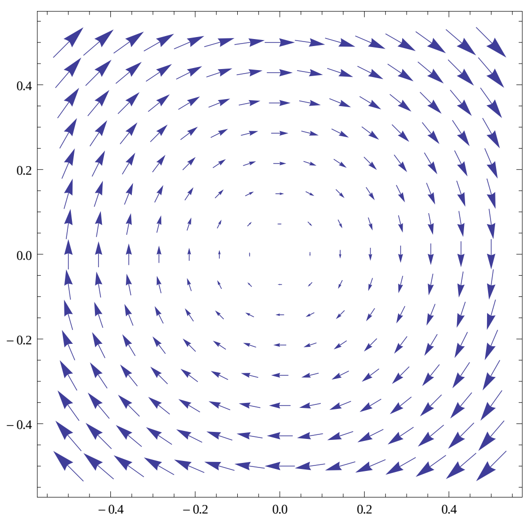
Some possibly useful formulas:

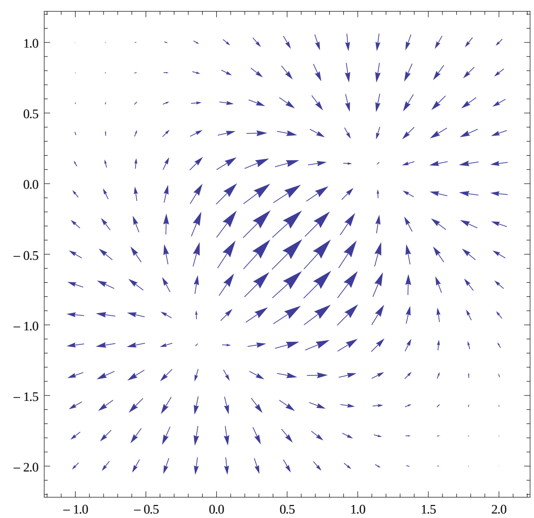
$$\cos^2 t = \frac{1}{2}(1 + \cos(2t))$$
$$\sin^2 t = \frac{1}{2}(1 - \cos(2t))$$

1. All but one of the following vector fields are gradient fields. Which one can't be a gradient field? **Why?**









2. Let $\mathbf{F}(x, y) = \langle ye^x + e^x + x^2 - 1, e^x + y^4 + \sin(\pi y)\rangle$.

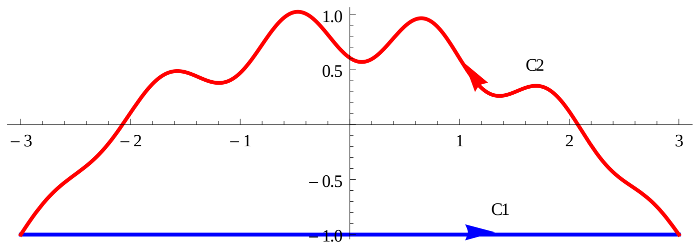
   (a) Find a potential function for $\mathbf{F}$.

   (b) Compute the flow of $\mathbf{F}$ along the part of the curve $y = \cos(\pi x)$ going from $(0, 1)$ to $(3, -1)$.

3. Find all sources and sinks of the vector field $\mathbf{F}(x, y) = \langle y^4, xy^3 e^x \rangle$.

4. Find all sources and sinks of the vector field $\mathbf{G}(x,y) = \left\langle \frac{x+y}{x^2+y^2}, \frac{y-x}{x^2+y^2} \right\rangle$.

5. Let $\mathbf{F}(x, y) = \langle e^x y^2, 2e^x y - 1 \rangle$. The curve C consists of two parts, C_1 and C_2, as shown below.



(a) Find the flow of $\mathbf{F}$ along C.

(b) Find the flow of $\mathbf{F}$ along C_1.

(c) Find the flow of $\mathbf{F}$ along C_2.

6. Let $\mathbf{F}(x, y) = \langle x + y, \frac{1}{2}y^2 \rangle$. Let C be the curve going from $(1, 0)$ to $(0, 2)$ along the parabola $4 - 4x = y^2$, then from $(0, 2)$ to $(-1, 0)$ along the parabola $4 + 4x = y^2$, then from $(-1, 0)$ to $(1, 0)$ along the x-axis. Let R be the region bounded by C.

   (a) Explain why you can measure the flow of $\mathbf{F}$ across C by the double integral

$$\iint_R (1 + y)\, dx\, dy.$$

   (b) Use the transformation $x = u^2 - v^2$ and $y = 2uv$ (assume $u, v \geq 0$) to compute the above integral. (The region is a little tricky.) Which direction is the flow of $\mathbf{F}$ across C?

7. Let R be the region bounded by the curves $x^2 - 4y^2 = 1$, $x^2 - 4y^2 = 9$, $x - 2y = 1$, and $x - 2y = 6$. Compute the area of R.

8. Compute $\iint_D (x+y)^2 \, dx \, dy$ where D is the unit disk.

9. Compute the area of the interior of the ellipse $4x^2 + 9y^2 = 1$ by two methods: Gauss-Green integration and 2D transformations.

10. Rewrite the following as one double integral by changing the order of integration (/direction of slices).

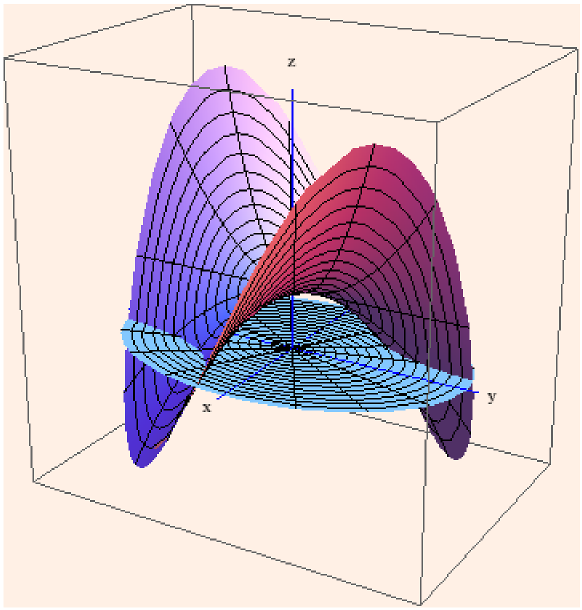
$$\int_{-2}^{-1}\int_{-x}^{2} f(x,y)\,dy\,dx + \int_{-1}^{1}\int_{1}^{2} f(x,y)\,dy\,dx + \int_{1}^{2}\int_{x}^{2} f(x,y)\,dy\,dx$$

11. Alice and Bob want to meet at the coffee shop to study for their upcoming exam. They agree that "about 2pm" is a good time. For these two, the probability distribution of their arrival time is given by $p(x, y) = 4(1 - x)(1 - y)$, where x measures the number of hours after 2pm that Alice arrives, and y measures the number of hours after 2pm that Bob arrives, both between 0 and 1 hour. (The distribution is zero outside of this range.)

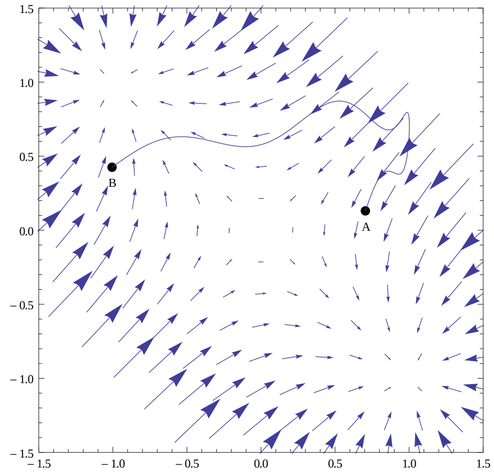
    Bob is impatient, and will leave if Alice isn't already there when he arrives. If Alice arrives and Bob isn't already there, she will wait for up to half an hour, after which she will leave.

    What is the probability that the two actually meet to study together?

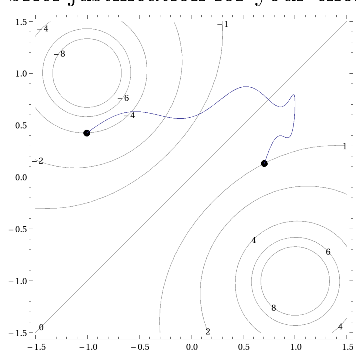
12. Below is shown the graph of a function $f(x, y)$ over a region R in the xy-plane. Without any computation, is $\iint_R f(x, y)\, dA$ positive, negative, or (nearly) zero? Why?

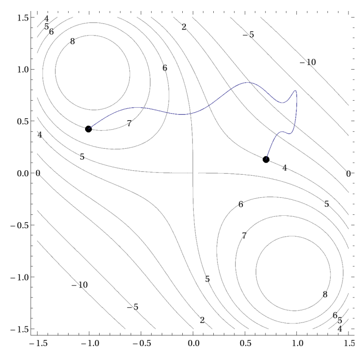


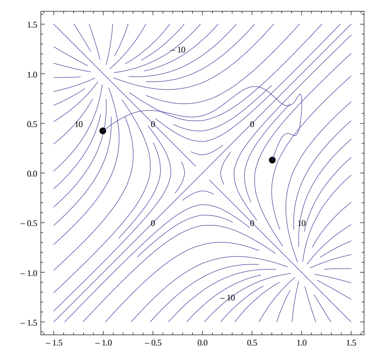
13. Below is shown part of a gradient field $\mathbf{F}$, together with a curve C.

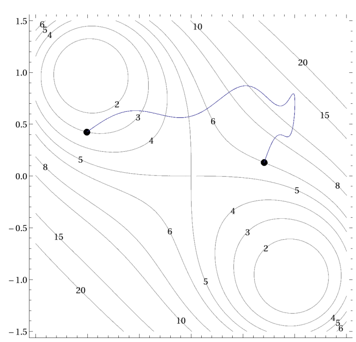


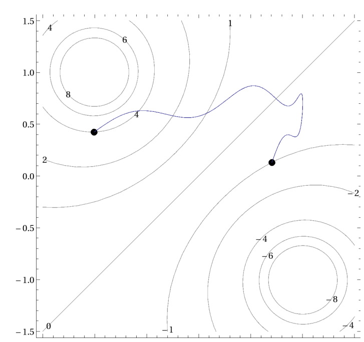
(a) (8 points) One of the following is the contour plot of a potential function for $\mathbf{F}$. Circle it. Give a brief justification for your choice. (The curve C is also shown in each.)











(b) (10 points) Find $\int_C \mathbf{F} \cdot \langle dx, dy \rangle$, assuming C is parametrized to go from A to B.

14. Find the centroid of the region bounded by the curves $y = x^2 - 16$ and $y = 4 - x^4$.