WORKSHOP 17: §8.4, 8.5, 8.9 MARCH 21, 2017

Name: ______________________________

Induction can be phrased in formal logic:

as a proposition:

$$P(1) \wedge \forall k\ (P(k) \to P(k+1)) \to \forall n\ P(n)$$

or as an argument:

$$\begin{array}{ll} & P(1) \\ & \forall k\ (P(k) \to P(k+1)) \\ \hline \therefore & \forall n\ P(n) \end{array}$$

(where the domain for all variables is $\mathbb{Z}^+$).

(1) Use Tables 1.12.1 and 1.13.1 (of common valid arguments) to prove that the following argument is valid. Explain why this suggests that the Principle of Mathematical Induction is reasonable.

$$\begin{array}{ll} & P(1) \\ & \forall k \in \mathbb{Z}^+\ (P(k) \to P(k+1)) \\ \hline \therefore & P(4) \end{array}$$

(2) Which of the following (variants of induction) are also correct? The domain for all variables is $\mathbb{Z}^+$, but the domains of quantification are modified by shorthand such as "$\forall k \geq 5$," which means the domain for k is $\{5, 6, 7, 8, \dots\}$.

You are not asked to prove your answers; the point is just to understand why induction "should" work and be able to extend this understanding to slightly different settings.

(a) $P(5) \wedge \forall k \geq 5\ (P(k) \to P(k+1)) \to \forall n \geq 5\ P(n)$
(b) $P(5) \wedge \forall k \geq 3\ (P(k) \to P(k+1)) \to \forall n \geq 3\ P(n)$
(c) $P(5) \wedge \forall k \geq 3\ (P(k) \to P(k+1)) \to \forall n \geq 5\ P(n)$
(d) $P(5) \wedge \forall k \geq 7\ (P(k) \to P(k+1)) \to \forall n \geq 5\ P(n)$
(e) $P(5) \wedge \forall k \geq 7\ (P(k) \to P(k+1)) \to \forall n \geq 7\ P(n)$

(3) Complete the following sketch of a proof by induction.

Theorem. *Define a sequence by $a_0 = 1$, $a_{n+1} = 2a_n + 1$ for $n \geq 0$. Then a closed form for $\{a_n\}$ is $a_n = 2^{n+1} - 1$.*

Proof. Proof by induction on n.

Base case, $n = 0$. $a_0 =$??? by definition, and $2^{???+1} - 1 =$???.

Inductive step. Assume that k is an integer, $k \geq$???, and $a_k =$?????. *[We want to show that $a_{k+1} = 2^{k+2} - 1$.]*

We have

$$\begin{aligned} a_{k+1} &= ????? && \text{by the recurrence relation} \\ &= 2 \cdot ????? + 1 && \text{the inductive hypothesis} \\ &= 2^{k+2} - 1 && \text{algebra} \end{aligned}$$

which is what we wanted to show.

So, by PMI *[the Principle of Mathematical Induction]*, the theorem holds. □

(4) Consider the proposition

$$P(0) \wedge \forall k \geq 0\ (P(k) \to P(k+1)) \to \forall n \geq 0\ P(n).$$

(a) If the domain for all variables is $\mathbb{N}$, then is this true?
(b) Explain why the proposition is not necessarily true if the domain is $\mathbb{Q}$.
(c) How might you prove the proposition with the domain $\mathbb{Q}$, using induction on something other than n? (There are many possibilities. Brainstorm and turn in as many as you can. Extra credit on HW9 for each nontrivially different idea you have.)