WORKSHOP 14: §7.3-7.4 MARCH 2, 2017

Name: ______________________________

(1) (a) What is the prime factorization of the product of $2^{10} \cdot 3^{22} \cdot 7^{100}$ and $2^{12} \cdot 3 \cdot 5^3$?
(b) In general, given the prime factorizations of two numbers, how do you find the prime factorization of their product?

(2) (a) Is $2^{21} \cdot 3^{77} \cdot 5^{12} \cdot 11^9$ a factor of $2^{42} \cdot 3^{96} \cdot 7^{19} \cdot 11^{12}$?
(b) In general, how can you tell from the prime factorizations of two numbers whether one is a factor of the other? Why does this work (reference 1b)?
(c) Use your answer to justify Theorem 7.3.2 from the zyBook.

(3) Here's a new proof that $\sqrt{2}$ is irrational.

Proof. Suppose to the contrary that $\sqrt{2}$ is rational. Then there are integers a, b such that $\sqrt{2} = a/b$ (and $b \neq 0$). Squaring and clearing denominators gives $a^2 = 2b^2$. *[So far, this is the same as the old proof, except we have not assumed that a/b is reduced.]* Consider the prime factorization of the number a^2, written with prime factors in nondecreasing order (not the multiplicity version). There are an even total number of primes in the factorization, because each prime in the factorization of a appears twice in a^2. Similarly, there are an even number of prime factors in b^2, and so an odd number of prime factors in $2b^2$. But since prime factorizations are unique, this contradicts that $a^2 = 2b^2$. □

(a) Does the same proof work to show that $\sqrt{3}, \sqrt{5}, \sqrt{7}$ are irrational (replacing 2 by 3, 5, 7)?
(b) Of course, $\sqrt{4}$ is rational. If you were to replace 2 by 4, where would the proof fail?
(c) For the same reason, the proof does not work to show that $\sqrt{6}$ is irrational; but $\sqrt{6}$ is indeed irrational. Modify the proof (focus on one prime's appearance instead of the total number of primes) to prove that $\sqrt{6}$ is irrational.

(4) Prove that in any three consecutive integers, exactly one of the three is divisible by 3.

(5) Prove that, except for (3,5,7), there are no three consecutive odd integers that are all prime.

(6) *Long gaps between primes.* Let $n \geq 2$ (and think of n as being very large). Explain why the $n - 1$ consecutive numbers $n! + 2, n! + 3, n! + 4, \ldots, n! + n$ are all composite.