WORKSHOP 10: §4.1-4.5 FEBRUARY 14, 2017 ♡

Name: ____

(1) Which of the following functions are one-to-one and which are onto?
  (a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$
  (b) $f : \mathbb{Z} \to \mathbb{N}$, $f(x) = |x|$
  (c) $f : \mathbb{N} \to \mathbb{N}$, $f(x) = |x|$
  (d) $f : \mathbb{N} \to \mathbb{Z}$, $f(x) = |x|$

(2) Prove or disprove the following statements. For proving statements about the floor/ceiling functions, use the following definition (a formal version of the textbook's definition):

$$\lfloor x \rfloor = n \qquad \text{if (and only if)} \qquad n \in \mathbb{Z} \ \wedge \ n \leq x < n+1$$

  (a) For any real numbers x and y, $\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$.
  (b) For any real number x and integer m, $\lfloor x + m \rfloor = \lfloor x \rfloor + m$.

(3) *Cardinalities and bijections*
  (a) For finite sets A and B, if there is a surjection $f : A \to B$, then what does this say about $|A|$ and $|B|$?
  (b) For finite sets A and B, if there is an injection $f : A \to B$, then what does this say about $|A|$ and $|B|$?
  (c) Using the previous two parts, if there is a bijection $f : A \to B$, then what does that say about $|A|$ and $|B|$?

Given the above, it makes sense to say that sets A and B (even infinite ones) "have the same cardinality" if there is a bijection $f : A \to B$.

  (d) Show that $\mathbb{N}$ and $\mathbb{Z}$ have the same cardinality. (That is, find a bijection from $\mathbb{N}$ to $\mathbb{Z}$. One way to do this is to "zip" together the positive and negative integers, alternating between them.)
  (e) Show that $\mathbb{Z}$ and $\{x : x \text{ is even}\}$ have the same cardinality.
  (f) Use the following table to show that $\mathbb{N}$ and $\mathbb{Q}^+$ have the same cardinality. (Follow the diagonals in the table and map elements of $\mathbb{N}$ as you go, skipping any members of $\mathbb{Q}^+$ that would be repeated this way. Why is the resulting function injective? Why is it surjective?)

$$\begin{array}{ccccc} 1/1 & 1/2 & 1/3 & 1/4 & \dots \\ 2/1 & 2/2 & 2/3 & 2/4 & \dots \\ 3/1 & 3/2 & 3/3 & 3/4 & \dots \\ 4/1 & 4/2 & 4/3 & 4/4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

  (g) After the previous problem, you should start to worry that maybe all infinite sets have the same cardinality (in which case the definition I just gave would be rather boring). Here's a proof that $\mathbb{N}$ and $\mathbb{R}^+$ do NOT have the same cardinality.

Proof. Suppose to the contrary that $\mathbb{N}$ and $\mathbb{R}^+$ have the same cardinality. Then there is some bijection $f : \mathbb{N} \to \mathbb{R}^+$. Such a bijection looks like the following:

$$\begin{aligned} 0 &\mapsto 1.523857102937\dots \\ 1 &\mapsto 128.129837641832\dots \\ 2 &\mapsto 0.000278381675\dots \\ 3 &\mapsto 23.998283999182\dots \\ &\vdots \end{aligned}$$

The above is just an example, so it cannot prove the result; but it will help to describe the rest of the argument.

It is convenient to forget about the integer parts and look instead at the function $g : \mathbb{N} \to [0, 1)$ given by

$$\begin{aligned} 0 &\mapsto 0.523857102937\ldots \\ 1 &\mapsto 0.129837641832\ldots \\ 2 &\mapsto 0.000278381675\ldots \\ 3 &\mapsto 0.998283999182\ldots \\ &\vdots \end{aligned}$$

Now g is not injective, unlike f. **Why not?** However, it is surjective. **Why?**
Now consider the real number x determined in the following fashion. The nth digit (of x) past the decimal is 0 unless the nth digit past the decimal in $g(n)$ is also zero, in which case the digit of x is 1. The important property here is that the nth digit is always different from the nth digit of $g(n)$. This means that x is not in the image of g (**why?**), contradicting that g is surjective. □