HOMEWORK 11: §10.1-10.6, 11.2 DUE APRIL 13

Name: ______________________________

- Please refer to the syllabus regarding allowed collaboration on this homework assignment.
- All answers should be fully justified.
- Your homework should be neatly written on additional paper; you may attach this cover page if you would like to keep the questions attached to the answers.

(1) The department is forming a committee to work out the details of a new course. The committee will consist of a president, vice president, and secretary, along with 7 at-large members (meaning they hold no special position, so are interchangeable). *(As usual in this chapter, your answers should not be simplified.)*
  (a) The department consists of only 10 faculty.
  (b) The department consists of 21 faculty.
  (c) The department consists of 21 faculty, but 8 of those are non-tenure-track and can only serve as at-large members.
  (d) The department consists of 21 (full) faculty, but Vi and Fi cannot hold the president/vice president roles together (at most one of them can have such a role).

(2) Consider the identity $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.
  (a) Prove the identity use the Binomial Theorem (zyBook Theorem 11.2.2)
  (b) Prove the identity with a *combinatorial proof*: find some situation for which the number of possibilities can be counted in two different ways, one giving rise to each side of the identity.
  (c) Prove the identity using induction; think about Pascal's Identity (zyBook Theorem 11.2.3). *(Hint: the easiest way to do this will involve a change of variables in a summation and splitting off the first/last term of a summation; see zyBook section 8.3. Also, for Pascal's Identity to hold for all k, we can put $\binom{n}{k} = 0$ whenever $k < 0$ or $k > n$.)*

(3) Identify what is **wrong** about the following attempts to count the surjective functions from $\{1, 2, 3, 4, 5\}$ to $\{u, v, w\}$. In general, a counting argument can fail in one of three ways:
  - some item that should be counted is not;
  - some item that should not be counted is; and/or
  - some item is counted more than once.

  To demonstrate that a counting argument is invalid, just find a specific example of one of the three bullets above.
  (a) Since the domain is larger than the target, every function is surjective; so there are 3^5 surjective functions.
  (b) Let $f(1) = u$, $f(2) = v$, $f(3) = w$; then $f(4), f(5)$ are free to be any elements of the target, so the number of surjective functions is 3^2.
  (c) There are 3! ways to have f send $\{1, 2, 3\}$ to $\{u, v, w\}$ bijectively; then 4, 5 can be sent anywhere, so there are $3! \cdot 3^2$ surjective functions.
  (d) Let x_u be an element such that $f(x_u) = u$, x_v such that $f(x_v) = v$, and x_w such that $f(x_w) = w$. Then $f(x_u), f(x_v), f(x_w)$ are determined, and since $x_i \neq x_j$ for $i \neq j$ (f is a function), the remaining 2 elements in the domain are free to be mapped anywhere. There are $P(5, 3)$ ways to choose $\{x_u, x_v, x_w\}$, then 3^2 ways to map the remaining domain elements. So the number of surjective functions is $P(5, 3) \cdot 3^2$.

  (As it turns out, counting surjective functions is not very easy using the tools from this course. They are counted by "Stirling functions of the second kind.")

"Anyone can count the seeds in an apple,
but no one can count the apples in a seed."