HOMEWORK 5: CHAPTER 3 DUE FEBRUARY 16
SOLUTIONS

The following comes up a couple of times in this homework; it doesn't appear as a theorem in the text, so we'll prove it here.

Theorem 1. *For any sets A, B, we have $A \cap B \subseteq A$ and $A \subseteq A \cup B$.*

Proof. These are analogues of the valid arguments Simplification and Addition (respectively). For the first: if $x \in A \cap B$, then $x \in A \wedge x \in B$, and Simplification implies that $x \in A$. For the second: if $x \in A$, then by Addition $x \in A \vee x \in B$, i.e. $x \in A \cup B$. $\square$

(1) Short exercises:
 (a) What is $\big|\{1,2,5\} \cup \{1,5,7\}\big|$?
 $= |\{1,2,5,7\}| = 4$
 (b) What is $\big|\{1,2,5\} \cap \{1,5,7\}\big|$?
 $= |\{1,2\}| = 2$
 (c) What is $\big|\{1,2,5\} \times \{1,5,7\}\big|$?
 $= |\{1,2,5\}| \cdot |\{1,5,7\}| = 3 \cdot 3 = 9$
 (d) What is $\mathcal{P}(\varnothing \times \{1,2,3\})$?
 $= \mathcal{P}(\varnothing) = \{\varnothing\}$

(2) If $A \subseteq B$, then what is $A \cup B$? What is $A \cap B$? Prove one of your answers (whichever you prefer).

Theorem. *If $A \subseteq B$, then $A \cup B = B$.*

Proof. We need to show two things: that $B \subseteq A \cup B$ and that $A \cup B \subseteq B$. The first one is true for any two sets A, B by Theorem 1.

For the second statement, let $x \in A \cup B$. Then $x \in A$ or $x \in B$. In the second case we are already done. In the first case, since $A \subseteq B$, $x \in B$ as well. *[Formally: the $\subseteq$ relation means $\forall y\ (y \in A \to y \in B)$, and we use universal instantiation and Modus Ponens to imply that $x \in B$.]* $\square$

Theorem. *If $A \subseteq B$, then $A \cap B = A$.*

Proof. **Claim 1: $A \cap B \subseteq A$.** This is actually true for any pair of sets A, B, by Theorem 1.

Claim 2: $A \subseteq A \cap B$. Let $x \in A$. Since $A \subseteq B$, also $x \in B$. Thus $x \in A \cap B$. $\square$

(3) Prove that if $A \subseteq B$, then $A \times C \subseteq B \times C$.

Proof. Let A, B, C be sets such that $A \subseteq B$. Let $x \in A \times C$. Then $x = (a, c)$ for some $a \in A$ and $c \in C$. Since $A \subseteq B$, $a \in B$ as well. But this means that $(a, c) \in B \times C$. $\square$

(4) Prove that if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Proof. Let A, B, C be sets such that $A \subseteq B$ and $B \subseteq C$. Let $x \in A$. Since $A \subseteq B$, we have $x \in B$. Now since $B \subseteq C$, we have $x \in C$. $\square$

(5) Prove or disprove the following.
 (a) For all sets A, B, if $A \subseteq B$, then $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Proof. Let A, B be sets such that $A \subseteq B$. Let $X \in \mathcal{P}(A)$. This means that $X \subseteq A$. Then Problem (4) implies that $X \subseteq B$. Thus $X \in \mathcal{P}(B)$. $\square$

(b) For all sets A, B, $\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$.

This is false. A counterexample: $A = \{1\}$, $B = \{2\}$. Then $\mathcal{P}(A \cup B) = \mathcal{P}(\{1, 2\}) = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$, but $\mathcal{P}(A) \cup \mathcal{P}(B) = \{\varnothing, \{1\}\} \cup \{\varnothing, \{2\}\} = \{\varnothing, \{1\}, \{2\}\}$.

(c) For all sets A, B, $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$.

Proof. Let A, B be sets. Let $X \in \mathcal{P}(A) \cup \mathcal{P}(B)$. That is, $X \in \mathcal{P}(A)$ or $X \in \mathcal{P}(B)$. WLOG, say $X \in \mathcal{P}(A)$ (otherwise switch A and B). That is, $X \subseteq A$. By Theorem 1, this implies that $X \subseteq A \cup B$, i.e. $X \in \mathcal{P}(A \cup B)$. □

(d) For all sets A, B, $\mathcal{P}(A \times B) = \mathcal{P}(A) \times \mathcal{P}(B)$.

This is false. In fact, they can never be the same: $\mathcal{P}(A \times B)$ is a set whose elements are sets, but $\mathcal{P}(A) \times \mathcal{P}(B)$ is a set whose elements are ordered pairs. For a concrete counterexample, let $A = \{a\}$, $B = \{b\}$. Then $\mathcal{P}(A \times B) = \mathcal{P}(\{(a, b)\}) = \{\varnothing, \{(a, b)\}\}$ but $\mathcal{P}(A) \times \mathcal{P}(B) = \{\varnothing, \{a\}\} \times \{\varnothing, \{b\}\} = \{(\varnothing, \varnothing), (\varnothing, \{b\}), (\{a\}, \varnothing), (\{a\}, \{b\})\}$.

(6) Suppose A, B are sets such that $A - B$, $B - A$, and $A \cap B$ are all nonempty. Prove that $A - B$, $B - A$, $A \cap B$ form a partition of $A \cup B$.

Proof. [To prove that something is a partition, you need to show that each part is nonempty, that the parts are pairwise disjoint, and that the union of the parts is the whole set.]

The three parts are nonempty by hypothesis.

Claim: $(A - B) \cap (B - A) = \varnothing$. Suppose to the contrary that $x \in A - B$ and $x \in B - A$. Since $x \in A - B$, $x \notin B$. Since $x \in B - A$, $x \in B$. These two facts contradict each other.

Claim: $(A - B) \cap (A \cap B) = \varnothing$. Suppose to the contrary that $x \in (A - B)$ and $x \in A \cap B$. The first statement implies $x \notin B$ and the second implies that $x \in B$, a contradiction.

Claim: $(B - A) \cap (A \cap B) = \varnothing$. This is the same statement as the last one except with A and B swapped.

We have finished the proof that the parts are pairwise disjoint.

Finally, we need to prove that $(A - B) \cup (B - A) \cup (A \cap B) = A \cup B$.

Claim: $(A - B) \cup (B - A) \cup (A \cap B) \subseteq A \cup B$. Let $x \in (A - B) \cup (B - A) \cup (A \cap B)$. Then x is in one of $A - B$, $A \cap B$, or $B - A$. In the first two cases, $x \in A$; in the last case, $x \in B$. In any case, $x \in A \cup B$.

Claim: $(A - B) \cup (B - A) \cup (A \cap B) \supseteq A \cup B$. Let $x \in A \cup B$.

Case $x \in A$: If $x \in B$, then $x \in A \cap B$. If instead $x \notin B$, then $x \in A - B$.

Case $x \in B$: If $x \in A$, then $x \in A \cap B$. If instead $x \notin A$, then $x \in B - A$. □

"Consider the set of all sets that have not yet been considered.
Oops, now it's empty..."