Name: Solutions

- **READ THE FOLLOWING DIRECTIONS!**
- **Do NOT open the exam until instructed to do so.**
- You have seventy-five (75) minutes to complete this exam. When you are told to stop writing, do it or you will lose all points on the page(s) you write on.
- You may not communicate with other students during this test.
- Keep your eyes on your own paper.
- No written materials of any kind are allowed. No scratch paper is allowed except as given by the proctor.
- No phones, calculators, or any other electronic devices are allowed for any reason, including checking the time (a simple wristwatch is fine).
- Any case of cheating will be taken extremely seriously.
- Show all your work and explain your answers when appropriate.
- Before turning in your exam, check to make certain you've answered all the questions.
- The tables of common logical equivalences and valid arguments appear on the back page of the exam.

Question	Points	Score
1	10	
2	12	
3	10	
4	10	
5	6	
6	6	
7	6	
8	8	
9	8	
10	8	
11	6	
12	8	
13	8	
Total:	106	

No justification is necessary on this page.

1. (10 points) Circle "Yes" for propositions and "No" for non-propositions.
   (a) Yes No "Is every function computable?" — question
   (b) Yes No "Every function is computable."
   (c) Yes No "Accessing the homework requires your Hawk account."
   (d) Yes No "Access the homework using your Hawk account." — command
   (e) Yes No "$x^2 = -1$." — predicate

2. (12 points) Circle "True" or "False" as appropriate.
   (a) True False $p \to \neg p$ is a contradiction. If $p = F$, then the proposition is T (vacuously)
   (b) True False $(p \leftrightarrow q) \vee (p \oplus q)$ is a tautology.
   (c) True False $n \mid 0$ is true for every integer n. $n \mid 0 \equiv \exists k \in \mathbb{Z}\,(0 = kn)$; this is true by taking $k=0$
   (d) True False $2 \subseteq \{1, 2, 3\}$. 2 is not a set and so is not a subset of anything.
   (e) True False $\varnothing \subseteq \{1, 2, 3\}$. The empty set is a subset of ANY set.
   (f) True False $\left|\mathcal{P}(\mathcal{P}(\varnothing))\right| = 2$. $|\mathcal{P}(\mathcal{P}(\varnothing))| = 2^{|\mathcal{P}(\varnothing)|} = 2^{2^{|\varnothing|}} = 2^{2^0} = 2^1 = 2$
   -or-, $\mathcal{P}(\mathcal{P}(\varnothing)) = \mathcal{P}(\{\varnothing\}) = \{\underset{1}{\varnothing}, \underset{2}{\{\varnothing\}}\}$

3. (10 points) The table below records the truth values of $P(x, y)$ where the domain for x is $\{x_1, x_2, x_3, x_4\}$ and the domain for y is $\{y_1, y_2, y_3, y_4\}$. For example, $P(x_2, y_3)$ is true by looking in the 2nd row and 3rd column; similarly, $P(x_2, y_4)$ is true.

$x \setminus y$	y_1	y_2	y_3	y_4
x_1	F	F	F	T
x_2	T	F	T	T
x_3	F	T	F	T
x_4	T	T	F	T

Determine whether each of the following is True or False.
   (a) True False $\exists x\, \exists y\, P(x, y)$ e.g., $x = x_1$ and $y = y_4$
   (b) True False $\exists x\, \forall y\, P(x, y)$ there is no row with all T
   (c) True False $\forall y\, \exists x\, P(x, y)$ every column has a T
   (d) True False $\exists y\, \forall x\, P(x, y)$ the column y_4 has all T
   (e) True False $\forall x\, \exists y\, P(x, y)$ every row has a T
   (f) True False $\forall x\, \forall y\, P(x, y)$ not every entry is T

4. (10 points) Circle "Yes" or "No" as appropriate. If the answer is Yes, nothing else is needed. If the answer is No, prove why.

(a) Yes **No**

Consider the set A of binary strings of length 4, and define $S_1 = \{x \in A : x \text{ starts with } 1\}$, $S_{00} = \{x \in A : x \text{ starts with } 00\}$. Do S_1, S_{00} form a partition of A?

The string 0100 is in A but neither S_1 nor S_{00}.

(So $S_1 \cup S_{00} \neq A$ as required of a partition.)

$S_1 \cap S_{00} = \emptyset$ though

Defining $S_{01} = \{x \in A: x \text{ starts with } 01\}$,
it is true that S_1, S_{00}, S_{01} form a partition.

(b) **Yes** No

Let $B = \{1,2,3,4\}$ and $C = \{3,4,5,6,7\}$, and let $D = B \times C$. For each $b \in B$, define $S_b = \{(b,x) : x \in C\}$. Do S_1, S_2, S_3, S_4 form a partition of D?

$S_1 = \{(1,3), (1,4), (1,5), (1,6), (1,7)\}$
$S_2 = \{(2,3), (2,4), (2,5), (2,6), (2,7)\}$
$S_3 = \{(3,3), (3,4), (3,5), (3,6), (3,7)\}$
$S_4 = \{(4,3), (4,4), (4,5), (4,6), (4,7)\}$

No S_i is empty.
The S_i are pairwise disjoint [if $i \neq j$, then $(i,x) \neq (j,y)$].
The union, $\bigcup_{i=1}^{4} S_i = D$.

(c) Yes **No**

For each $n \in \mathbb{N}$, let $S_n = \{q \in \mathbb{Q} : \exists b\ (b \neq 0 \wedge q = n/b)\}$. Do $S_0, S_1, S_2, S_3, \ldots$ form a partition of $\mathbb{Q}$?

$\frac{5}{3} \in S_5$, but also $\frac{5}{3} = \frac{10}{6} \in S_{10}$, so $S_5 \cap S_{10} \neq \emptyset$.

(I.e., the sets are not pairwise disjoint, as required of a partition.)

They do cover all of $\mathbb{Q}$: $\bigcup_{i=0}^{\infty} S_i = \mathbb{Q}$.

5. (6 points) Let $p =$ "Alice is smart," $q =$ "Alice studies," and $r =$ "Alice gets an A on the exam." Translate the proposition

"Alice is smart, but if she doesn't study, she won't get an A on the exam."

into formal logic, using the propositional variables p, q, r and any connectives.

$p \wedge (\neg q \rightarrow \neg r)$

6. (6 points) Let $P(x) =$ "x is smart," $Q(x) =$ "x studies," and $F(x, y) =$ "x and y are friends." Translate the proposition

"Everyone who is smart has a friend who studies"

into formal logic.

$\forall x \left(P(x) \longrightarrow \exists y \left(F(x,y) \wedge Q(y) \right) \right)$

7. (6 points) Simplify $\neg(\forall x\ (x = 0 \lor \exists y\ xy \geq 1))$ to a logically equivalent proposition that does not use the negation symbol.

$$\equiv \exists x\ (x \neq 0 \land \forall y\ xy < 1)$$

8. (8 points) Rewrite the proposition $p \leftrightarrow q$ using only the connectives $\neg, \land$. (That is, find another proposition that is logically equivalent to the given one, but uses only the symbols $(,), p, q, \neg, \land$.) Justify your answer.

$\equiv (p \to q) \land (q \to p)$ Cond. Law

$\equiv (\neg p \lor q) \land (\neg q \lor p)$ Cond. Law

$\equiv \neg(p \land \neg q) \land \neg(q \land \neg p)$ DeMorgan & Double-negative

9. (8 points) Prove, using the table of common valid arguments, that the following argument is valid. (It may help to rewrite some of the statements more formally first.)

	Every student who gets an A on both the homework and the quiz did all the reading.	(a)
	Olive is a student.	(b)
	Olive did not do all the reading.	(c)
	Olive got an A on the homework.	(d)
$\therefore$	Olive did not get an A on the quiz.	

(1) If Olive gets an A on both HW & Quiz, then she did all the reading. Univ. Inst. (a), (b)

(2) Olive did not get an A on both the HW & Quiz. Modus Tollens, (1), (c)

(3) Olive did not get A on HW or she did not get A on Quiz. DeMorgan, (2)

(4) Olive did not get an A on the Quiz. Disj. Syll. (3), (d)

10. (8 points) Prove that if $7n - 3$ is even, then n is odd.

We prove the contrapositive.

Let n be even. So $n = 2k$ for some $k \in \mathbb{Z}$.

Then $7n - 3 = 14k - 3$

$= 2(7k - 2) + 1$.

Since $k \in \mathbb{Z}$, also $7k - 2 \in \mathbb{Z}$, so $7n - 3$ is odd. $\square$

11. (6 points) Prove by contradiction: in every class of 30 students, there is some month in which at least three students have a birthday.

Suppose to the contrary that in some class of 30, every month has ≤ 2 birthdays.

Then $\#\text{students} = \#\text{birthdays} \leq 2 \cdot 12 = 24 < 30$,

contradicting that $\#\text{students} = 30$. $\square$

12. (8 points) Prove that for every integer n, $n^2 + 3n + 1$ is odd.

Let n be an integer.

Case 1: n is even. So $n = 2k$ for some $k \in \mathbb{Z}$.

Then $n^2 + 3n + 1 = 4k^2 + 6k + 1$

$= 2(2k^2 + 3k) + 1.$

$k \in \mathbb{Z} \rightarrow 2k^2 + 3k \in \mathbb{Z}$, so $n^2 + 3n + 1$ is odd.

Case 2: n is odd. So $n = 2k+1$ for some $k \in \mathbb{Z}$.

Then $n^2 + 3n + 1 = 4k^2 + 4k + 1 + 6k + 3 + 1$

$= 4k^2 + 10k + 5$

$= 2(2k^2 + 5k + 2) + 1.$

$k \in \mathbb{Z} \rightarrow 2k^2 + 5k + 2 \in \mathbb{Z}$, so $n^2 + 3n + 1$ is odd.

$\square$

13. (8 points) Prove that for all sets A, B, $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.

You may use the following theorems (we proved most of them in class), but indicate clearly where you use them. (You probably will not need to use all of them.)

1. For all sets X, Y, $X \cap Y \subseteq X$.
2. For all sets X, Y, $X \subseteq X \cup Y$.
3. If $X \subseteq Y$ and $X \subseteq Z$, then $X \subseteq Y \cap Z$.
4. If $X \subseteq Y \cap Z$, then $X \subseteq Y$.
5. If $X \subseteq Y$, then $\mathcal{P}(X) \subseteq \mathcal{P}(Y)$.

We need to show that each set is a subset of the other.

$\subseteq$] By (1), $A \cap B \subseteq A$.
By (5), $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A)$.

Similarly, by (1) $A \cap B \subseteq B$
so by (5) $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(B)$.

Combining & with (3), $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$,
as desired.

$\supseteq$] Let $X \in \mathcal{P}(A) \cap \mathcal{P}(B)$.
Then $X \in \mathcal{P}(A)$ and $X \in \mathcal{P}(B)$, [def. of $\cap$]
i.e. $X \subseteq A$ and $X \subseteq B$. [def. of $\mathcal{P}$]

By (3), $X \subseteq A \cap B$,
hence $X \in \mathcal{P}(A \cap B)$.

$\square$

Alternative for $\subseteq$] Let $X \in \mathcal{P}(A \cap B)$. Then $X \subseteq A \cap B$ [def of $\mathcal{P}$]
By (4), $X \subseteq A$ and $X \subseteq B$.
So $X \in \mathcal{P}(A)$ and $X \in \mathcal{P}(B)$ [def of $\mathcal{P}$].

Then $X \in \mathcal{P}(A) \cap \mathcal{P}(B)$ [def of $\cap$].

Scratch Paper - Do Not Remove