Name: Solutions

- **READ THE FOLLOWING DIRECTIONS!**
- **Do NOT open the exam until instructed to do so.**
- You have 75 minutes to complete this exam. When you are told to stop writing, do it or you will lose all points on the page you write on.
- You may not communicate with other students during this test.
- No written materials of any kind are allowed. No scratch paper is allowed except as given by the proctors.
- No phones, calculators, or any other electronic devices are allowed for any reason, including checking the time (a simple wristwatch is fine).
- Any case of cheating will be taken extremely seriously.
- Show all your work and explain your answers.
- Before turning in your exam, check to make certain you've answered all the questions.

| Question: | 1 | 2 | 3 | 4 | 5 | 6 | Total |
|---|---|---|---|---|---|---|---|
| Points: | 15 | 18 | 14 | 18 | 15 | 10 | 90 |
| Score: | | | | | | | |

1. (15 points) Consider the function

$$f(x,y) = \begin{cases} \dfrac{7x^3y}{2x^4+y^4} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Where is $f$ continuous?

Everywhere other than $(0,0)$, $f$ is a rational function $[2x^4+y^4=0$ only if $(x,y)=(0,0)]$,

so it is continuous.

At $(0,0)$, we need to ask: is $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$ ?

Along $x$-axis, $\lim_{x\to 0} f(x,0) = \lim_{x\to 0} \frac{0}{2x^4} = 0.$

Along $y=x$, $\lim_{x\to 0} f(x,x) = \lim_{x\to 0} \frac{7x^4}{3x^4} = \frac{7}{3}.$

So $\lim_{(x,y)\to(0,0)} f(x,y)$ DNE,

and hence $f$ is not continuous at $(0,0)$.

2. Let $f(x, y) = x^2 + xy^2 + 2y^2$.

   (a) (10 points) Compute each of the following:

   i. $f_x$ $= 2x + y^2$

   ii. $f_y$ $= 2xy + 4y$

   iii. $f_{xx}$ $= 2$

   iv. $f_{xy}$ $= 2y$

   v. $f_{yy}$ $= 2x + 4$

   (b) (8 points) Find and classify the critical points of $f$.

$$\begin{cases} 2x + y^2 = 0 \\ 2xy + 4y = 0 \end{cases} \longrightarrow 2y(x+2) = 0 \Longrightarrow \begin{array}{c} y = 0 \\ x = 0 \end{array} \quad \text{OR} \quad \begin{array}{c} x = -2 \\ y = \pm 2 \end{array}$$

$$D = \begin{vmatrix} 2 & 2y \\ 2y & 2x+4 \end{vmatrix} = (4x+8) - 4y^2$$

$D(0,0) = 8 > 0$, $f_{xx}(0,0) = 2 > 0$, so

$D(-2, \pm 2) = -16 < 0$, so

$$\boxed{\begin{array}{l} (0,0) \text{ is a local min} \\ (-2, \pm 2) \text{ are saddle points} \end{array}}$$

3. At right is the temperature $T$ (in Celsius) of the surface of Planet X at coordinates $(x, y)$ at several points.

   A rover is traveling along the path $x(t) = 2t^2$, $y(t) = t^3 - t$, where $t$ is the time in hours since it landed.

| $T$ \ $y$ | | | | | |
|---|---|---|---|---|---|
| 3 | 2 | 4 | 5 | 6 | 2 |
| 2 | 3 | 5 | 6 | 8 | 3 |
| 1 | 1 | 4 | 3 | 7 | 3 |
| 0 | 3 | 2 | 4 | 6 | 3 |
| −1 | 6 | 8 | 5 | 4 | 1 |
| $x$ | 0 | 1 | 2 | 3 | 4 |

(a) (6 points) Estimate $T_x(2, 0)$ and $T_y(2, 0)$.
Show enough work that I know that you know what you are doing.

$$T_x(2,0) = \lim_{h\to 0} \frac{T(2+h,0) - T(2)}{h} \approx \frac{T(3,0) - T(2,0)}{3-2} = \frac{6-4}{1} = 2.$$

$$T_y(2,0) \approx \frac{T(2,1) - T(2,0)}{1} = \frac{3-4}{1} = -1$$

(b) (4 points) Write the Chain Rule for computing $\dfrac{dT}{dt}$.

$$\frac{dT}{dt} = \frac{\partial T}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial T}{\partial y}\cdot\frac{dy}{dt}$$

(c) (4 points) Estimate $\dfrac{dT}{dt}$ one hour into the rover's trip. What are the units?

$t=1$ $\quad x(1)=2,$
$y(1)=0$

$$\frac{dT}{dt}(1) = \frac{\partial T}{\partial x}(2,0)\cdot\frac{dx}{dt}(1) + \frac{\partial T}{\partial y}(2,0)\cdot\frac{dy}{dt}(1)$$

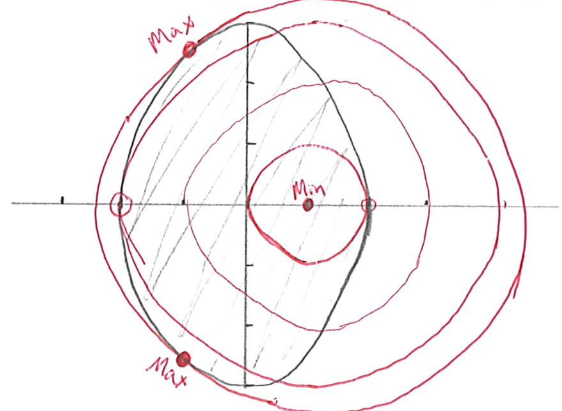
$$\approx 2\cdot\left(4t\big|_{t=1}\right) - 1\cdot\left(3t-1\big|_{t=1}\right) = 2\cdot 4 - 1\cdot 2 = \boxed{6\ \frac{°C}{hr}}.$$

4. Let $f(x,y) = (x-1)^2 + y^2$, and let $D$ be the solid ellipse defined by $9x^2 + 4y^2 \le 36$.

(a) (5 points) Can you guarantee that $f$ attains both a maximum and minimum on $D$? Explain.

Yes: $f$ is continuous on $D$ (in fact, on all of $\mathbb{R}^2$), & $D$ is closed & bounded, so the Extreme Value Theorem applies.

(b) (5 points) Sketch $D$ together with a contour plot of $f$, and use this to estimate the locations of the maximum and minimum of $f$ on $D$ (if they exist). (Indicate these locations in your drawing.)




(c) (8 points) Find the precise maximum and minimum of $f$ on $D$, or say that they do not exist.

Interior: $\begin{cases} 2(x-1)=0 \\ 2y=0 \end{cases} \Rightarrow \begin{matrix} x=1 \\ y=0 \end{matrix}$

Boundary: $\begin{cases} 2(x-1) = \lambda(18x) \\ 2y = \lambda(8y) \Rightarrow \lambda = \frac{1}{4} \text{ OR } y=0 \\ 9x^2+4y^2=36 \end{cases}$

$\lambda = \frac{1}{4} \Downarrow x = -\frac{4}{5} \Downarrow y = \pm\frac{3}{5}\sqrt{21}$

$y=0 \Downarrow x = \pm 2$

$f(1,0) = 0$ Min

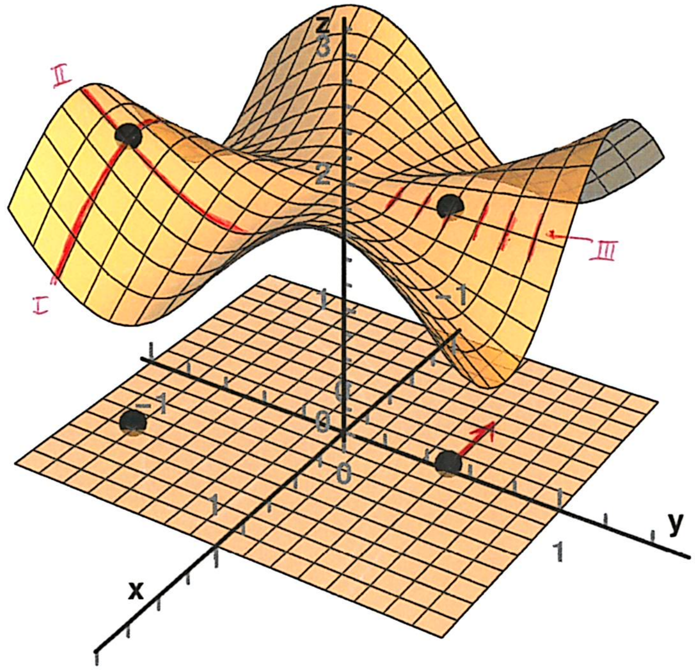
$f(2,0) = 1$

$f(-2,0) = 9$

$f\left(-\frac{4}{5}, \pm\frac{3}{5}\sqrt{21}\right) = \left(-\frac{9}{5}\right)^2 + \left(\pm\frac{3}{5}\sqrt{21}\right)^2$

$= \frac{81}{25} + \frac{189}{25} = \frac{270}{25} = 10.8$ Max

5. Below is shown the graph of a function $f$, the points $P = (0.5, -0.75)$ and $Q = (0, 0.5)$ in the $xy$-plane, and the lifts of those points to the graph.




$\nabla f$ is $\perp$ to level curves, & here $y$-axis is a level curve; & is in direction of increase

OR $f_x(Q) < 0$, $f_y(Q) = 0$

(a) (6 points) Sketch $\nabla f(Q)$ (in the $xy$-plane, in the picture above).

(b) (9 points) Circle below the sign of each partial derivative. Briefly justify each.

$f_{xx}(P)$ : + − 0 curve I is concave down

$f_{yy}(P)$ : + − 0 curve II is... well, to the right, it looks concave up (so +?) but to the left it doesn't seem very concave (so 0?)

OR

$f_{xy}(Q)$ : + − 0 as $y$ increases, the slopes $f_x$ (in III) are $\underbrace{\text{becoming more negative}}$, i.e. decreasing, so $(f_x)_y < 0$.

6. (10 points) Circle 'True' or 'False' and give a brief justification.

(a) True **False** If $\nabla f(a,b)$ exists, then $f$ is differentiable at $(a,b)$.

Our definition of "differentiable" is roughly that the tangent plane is a good approximation to the graph. It is not enough just for $f_x$ & $f_y$ to exist [see Workshop 4].

(b) True **False** The tangent plane to graph of $f(x,y)$ at a point $(a,b,f(a,b))$ must contain all the tangent lines to the graph at $(a,b,f(a,b))$.

True if $f$ is differentiable at $(a,b)$.
[See Workshop 4.]

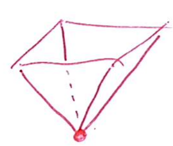
(c) **True** False If $f$ is differentiable and has a local maximum at $(a,b)$, then $\nabla f(a,b) = \mathbf{0}$.

Local maxima occur at critical points.

(d) **True** False If $f$ is differentiable at $(a,b)$, then it is continuous at $(a,b)$.

If the tangent plane is close to the graph, then the graph cannot have a jump or a "crease".

(e) True **False** If $f$ is continuous at $(a,b)$, then it is differentiable at $(a,b)$.

e.g. 

**Scratch Paper - Do Not Remove**