Name: Solutions

- **READ THE FOLLOWING DIRECTIONS!**
- **Do NOT open the exam until instructed to do so.**
- You have 75 minutes to complete this exam. When you are told to stop writing, do it or you will lose all points on the page you write on.
- You may not communicate with other students during this test.
- No written materials of any kind are allowed. No scratch paper is allowed except as given by the proctors.
- No phones, calculators, or any other electronic devices are allowed for any reason, including checking the time (a simple wristwatch is fine).
- Any case of cheating will be taken extremely seriously.
- Show all your work and explain your answers.
- Before turning in your exam, check to make certain you've answered all the questions.

Question:	1	2	3	4	5	6	Total
Points:	12	15	32	10	6	10	85
Score:							

1. (12 points) Let $\mathbf{u} = \langle 5, -2, 4, 7 \rangle$ and $\mathbf{v} = \langle 1, 0, 3, -1 \rangle$.

(a) Compute $\mathbf{u} - 2\mathbf{v}$.

$= \langle 5, -2, 4, 7 \rangle - \langle 2, 0, 6, -2 \rangle$

$= \langle 3, -2, -2, 9 \rangle$.

(b) Compute $\mathbf{u} \cdot \mathbf{v}$.

$= 5 \cdot 1 + (-2) \cdot 0 + 4(3) + 7(-1)$

$= 5 + 0 + 12 - 7$

$= 10$.

(c) Compute $\|7\mathbf{v}\|$.

$= 7\|v\| = 7\sqrt{1^2 + 0^2 + 3^2 + (-1)^2} = 7\sqrt{11}$.

(d) Compute $\text{proj}_{\mathbf{v}}\, \mathbf{u}$.

$= \frac{u \cdot v}{v \cdot v} v = \frac{10}{11} \langle 1, 0, 3, -1 \rangle$.

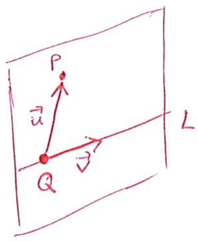
2. Consider the line L parametrized by $\ell(t) = \underbrace{\langle 1, 2, -1\rangle}_{Q} + t\underbrace{\langle -3, 2, 1\rangle}_{\vec{v}}$ and the point $P = (5, -1, 1)$.

(a) (3 points) Is P on L?

i.e., is there a solution to $\ell(t) = P \longleftrightarrow \begin{cases} 1 - 3t = 5 \\ 2 + 2t = -1 \\ -1 + t = 1 \Rightarrow t = 2 \end{cases} \longrightarrow 6 = -1$ ↯

No.

(b) (8 points) Find the Cartesian equation of a plane containing both P and L.



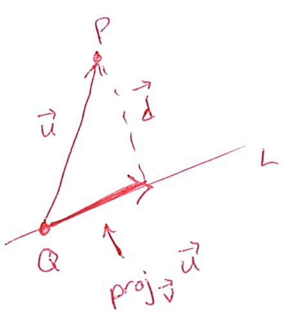
$\vec{u} = \vec{QP} = \langle 5, -1, 1\rangle - \langle 1, 2, -1\rangle = \langle 4, -3, 2\rangle$

$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -3 & 2 \\ -3 & 2 & 1 \end{vmatrix} = \langle -3-4, -(4+6), 8-9\rangle = \langle -7, -10, -1\rangle$

Take $\vec{n} = \langle +7, +10, +1\rangle$.

$$\boxed{7(x-1) + 10(y-2) + (z+1) = 0}$$

(c) (4 points) Find the distance from P to L. *(No need to simplify.)*



$\|\text{proj}_{v} u\| = |\text{comp}_v u|$

$= \dfrac{|u \cdot v|}{\|v\|}$

$= \dfrac{|-12-6+2|}{\sqrt{9+4+1}}$

$= \dfrac{16}{\sqrt{14}}$

$\|\vec{d}\| = \sqrt{\|u\|^2 - \|\text{proj}_v u\|^2}$

$= \sqrt{(16+9+4) - \dfrac{16^2}{14}}$

OR

$\vec{d} = \vec{u} - \text{proj}_{\vec{v}} \vec{u}$

$= \langle 4, -3, 2\rangle + \dfrac{16}{14}\langle -3, 2, 1\rangle$

$= \dfrac{1}{7}\left(\langle 28, -21, 14\rangle + \langle -24, 16, 8\rangle\right)$

$= \dfrac{1}{7}\langle 4, -5, 22\rangle$

$\|\vec{d}\| = \dfrac{1}{7}\sqrt{16 + 25 + 484}$

3. Consider the vector equation $\langle t, t^2, \frac{2}{3}t^3 \rangle$. *(This problem continues onto the next page.)*

(a) (3 points) Find $\mathbf{r}'(t)$.

$\langle 1, 2t, 2t^2 \rangle$

(b) (3 points) Find $\mathbf{r}''(t)$.

$\langle 0, 2, 4t \rangle$

(c) (3 points) Find (and simplify) $\|\mathbf{r}'(t)\|$.

$= \sqrt{1 + 4t^2 + 4t^4} = \sqrt{(1+2t^2)^2} = |1+2t^2| = 1+2t^2$

(d) (8 points) Find an equation for the tangent line to the curve at $t = 1$.

point $= \vec{r}(1) = (1, 1, \frac{2}{3})$

dir. vector $= \vec{r}'(1) = \langle 1, 2, 2 \rangle$

$\ell(t) = (1, 1, \frac{2}{3}) + t\langle 1, 2, 2 \rangle$.

(e) (5 points) Find the arc length of the part of the curve with $0 \le t \le 2$.

$$= \int_0^2 \|r'(t)\|\,dt = \int_0^2 (1+2t^2)\,dt$$

$$= \left[t + \frac{2}{3}t^3\right]_0^2$$

$$= 2 + \frac{16}{3} - 0$$

$$= \frac{22}{3}.$$

This is a continuation of the previous problem.

$r' = \langle 1, 2t, 2t^2 \rangle$ $\qquad$ $r'' = \langle 0, 2, 4t \rangle$

$\|r'\| = 1 + 2t^2$

(f) (6 points) Compute $\widehat{\mathbf{T}}(1)$, $\widehat{\mathbf{N}}(1)$, and $\widehat{\mathbf{B}}(1)$.

$$\hat{T}(t) = \frac{r'}{\|r'\|} = \frac{\langle 1, 2t, 2t^2 \rangle}{1+2t^2} \qquad \hat{T}(1) = \frac{\langle 1, 2, 2 \rangle}{3}$$

$$\hat{T}'(t) = \left(\frac{1}{1+2t^2}\right)' \langle 1, 2t, 2t^2 \rangle + \frac{\langle 1, 2t, 2t^2 \rangle'}{1+2t^2}$$

$$= -\frac{4t}{(1+2t^2)^2} \langle 1, 2t, 2t^2 \rangle + \frac{\langle 0, 2, 4t \rangle}{1+2t^2}$$

$$\hat{T}'(1) = -\frac{4}{9} \langle 1, 2, 2 \rangle + \frac{1}{3} \langle 0, 2, 4 \rangle$$

$$= \frac{1}{9} \left(\langle -4, -8, -8 \rangle + \langle 0, 6, 12 \rangle \right)$$

$$= \frac{1}{9} \langle -4, -2, 4 \rangle$$

$$= \frac{2}{9} \langle -2, -1, 2 \rangle.$$

$$\|\hat{T}'(1)\| = \frac{2}{9} \cdot 3 \qquad \hat{N}(1) = \frac{\hat{T}'(1)}{\|\hat{T}'(1)\|} = \frac{\langle -2, -1, 2 \rangle}{3}$$

$$\hat{B}(1) = \hat{T}(1) \times \hat{N}(1) =$$

$$= \frac{1}{3} \cdot \frac{1}{3} \cdot \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 2 \\ -2 & -1 & 2 \end{vmatrix} = \frac{1}{9} \langle 6, -6, 3 \rangle$$

$$\hat{B}(1) = \frac{\langle 2, -2, 1 \rangle}{3}$$

(g) (4 points) Compute $\kappa(1)$.

$$\kappa(1) = \frac{\|r'(1) \times r''(1)\|}{\|r'(1)\|^3}$$

$$r'(1) \times r''(1) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 2 \\ 0 & 2 & 4 \end{vmatrix} = \langle 4, -4, 2 \rangle$$

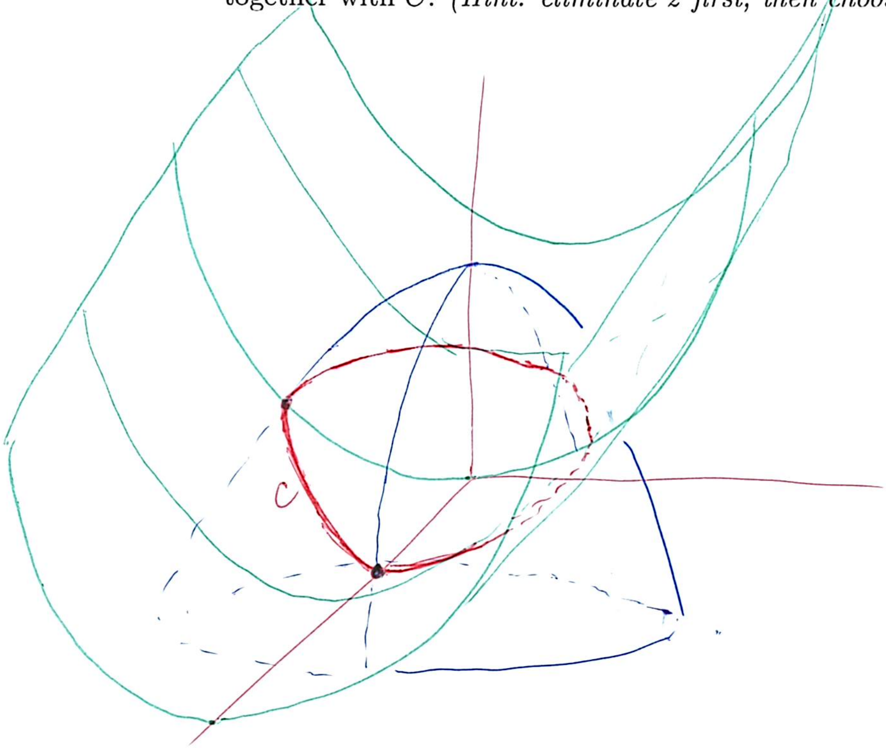
$$\| \cdots \| = \sqrt{16+16+4} = \sqrt{36} = 6$$

$$\|r'(1)\| = 3$$

$$\kappa(1) = \frac{6}{3^3} = \frac{2}{9}$$

parabolic cylinder paraboloid, opening downward

4. (10 points) Consider the two surfaces with equations $z = 3y^2$ and $x^2 + y^2 + z = 1$. Let C be the curve that is their intersection. Find a parametrization of C, and sketch the surfaces together with C. *(Hint: eliminate z first, then choose x and y.)*



$$x^2+y^2+(3y^2)=1$$

$$x^2+4y^2=1$$

$$\begin{cases} x = \cos t \\ y = \frac{1}{2}\sin t \\ z = \frac{3}{4}\sin^2 t \quad \left(= 1-\cos^2 t - \frac{1}{4}\sin^2 t\right) \end{cases}$$

$t \in [0, 2\pi]$

5. (6 points) Below is shown a trajectory of a particle (in the plane) together with the velocity at each of several points. Which direction is the acceleration at P? Give brief justifications. *(circle one answer in each row)*

(a) in the page   out of the page toward you   into the page away from you

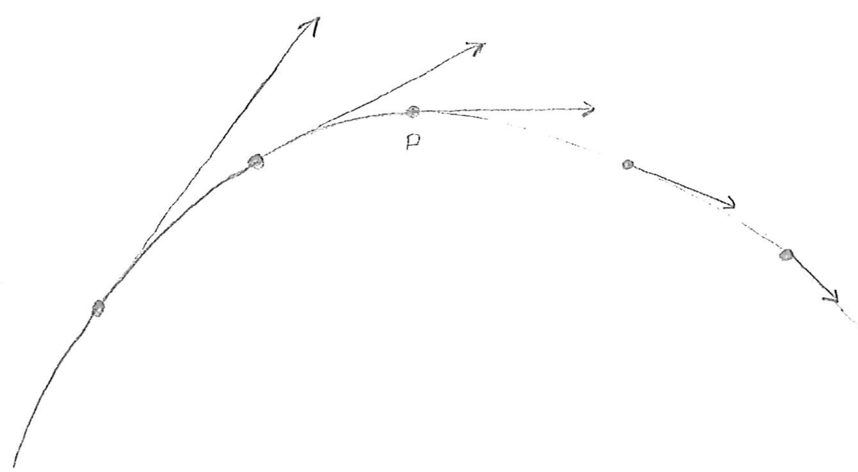
$\vec{a}$ only acts in the $\hat{T}$ & $\hat{N}$ directions, not $\hat{B}$ — OR — If $\vec{a}$ were into/out of the page, then the particle couldn't stay in the page.

(b) vertical   to the left   to the right

particle is slowing down ($\|v\|$ is decreasing), so $\vec{a}$ is opposite direction of motion

(c) horizontal   upward   downward

particle is turning downward



6. (10 points) Circle 'True' or 'False' and give a brief justification.

(a) True False For every scalar function $f(t)$ and every vector function $\mathbf{r}(t)$, we have $\frac{d}{dt}\Big(f(t)\,\mathbf{r}(t)\Big) = f(t)\,\mathbf{r}'(t)$.

$$= f'(t)\,\vec{r}(t) + f(t)\,\vec{r}\,'(t)$$

(b) True False For a curve in the plane of the paper, $\widehat{\mathbf{B}}$ always points out of the page toward you (when it exists).

may be into the page away from you.

e.g.,

(c) True False Consider a (finite) curve C in $\mathbb{R}^3$ and its projection/shadow P in the xy-plane. The arc length of C must be at least as large as the arc length of P.

If $\vec{r}_C = \langle x, y, z\rangle$, then $\vec{r}_P = \langle x, y, 0\rangle$

$\vec{r}_C\,' = \langle x', y', z'\rangle$ & $\vec{r}_P\,' = \langle x', y', 0\rangle$

$\|r_C'\| = \sqrt{(x')^2 + (y')^2 + (z')^2} \geq \|r_P'\| = \sqrt{(x')^2 + (y')^2 + 0}$

arc length of C $= \int \|r_C'\|\,dt \geq \int \|r_P'\|\,dt =$ arc length of P

(d) True False Consider a (finite) curve C in $\mathbb{R}^3$ and its projection/shadow P in the xy-plane. The curvature of C (at a point (x, y, z)) must be at least as large as the curvature of P (at the corresponding point $(x, y, 0)$).

E.g., the helix C vs. the circle P