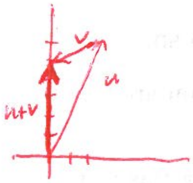
Name: ______________________________

- READ THE FOLLOWING DIRECTIONS!
- Do NOT open the exam until instructed to do so.
- You have 75 minutes to complete this exam. When you are told to stop writing, do it or you will lose all points on the page you write on.
- You may not communicate with other students during this test.
- No written materials of any kind are allowed. No scratch paper is allowed except as given by the proctors.
- No phones, calculators, or any other electronic devices are allowed for any reason, including checking the time (a simple wristwatch is fine).
- Any case of cheating will be taken extremely seriously.
- Show all your work and explain your answers.
- Before turning in your exam, check to make certain you've answered all the questions.

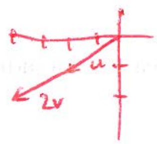
1. Let $\mathsf{u} = \langle 2, 6\rangle$ and $\mathsf{v} = \langle -2, -1\rangle$. Compute and plot the following together with u and v.

(a) $\mathsf{u} + \mathsf{v}$

$= \langle 0, 5\rangle$



(b) $2\mathsf{v}$ $= \langle -4, -2\rangle$



(c) the angle between u and v

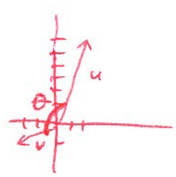
$u \cdot v = 2(-2) + 6(-1) = -10 = |u||v| \cos\theta$

$|u| = \sqrt{4+36} = \sqrt{40} = 2\sqrt{10}$

$|v| = \sqrt{4+1} = \sqrt{5}$

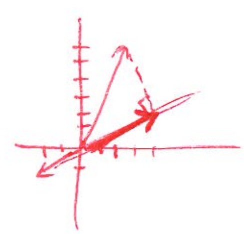
$\cos\theta = \frac{-10}{10\sqrt{2}} = -\frac{\sqrt{2}}{2}$

$\Rightarrow \theta = \frac{3\pi}{4}$



(d) the projection of u in the direction of v

$= \frac{u \cdot v}{v \cdot v} v = \frac{-10}{5} v = \langle 4, 2\rangle$



(e) If u and v live in the plane of this paper, and you consider the paper in the 3D classroom, then which direction does $\mathsf{u} \times \mathsf{v}$ point?

upward out of the paper

2. Consider the lines $\ell_1(t) = (0,1,3) + t(-2,-2,-10)$ and $\ell_2(t) = (-5,2,3)+t(-9,9,30)$. Are they parallel, perpendicular, or neither? Do they intersect or not? If they intersect or are parallel, find an xyz-equation of the plane containing them. Otherwise find the distance between them.

$\boxed{\text{Neither parallel nor perpendicular}}$: $(-2,-2,-10) \neq c(-9,9,30)$ [not //]
& $(-2,-2,-10)\cdot(-9,9,30) \neq 0$ [not ⊥]

Intersect? $\begin{cases} -2t = -5-9s \\ 1-2t = 2+9s \\ 3-10t = 3+30s \end{cases}$ $\xrightarrow{\text{substitute}}$ $1-5-9s = 2+9s$

$-6 = 18s$

$-\frac{1}{3} = s \overset{①}{\Rightarrow} t=1$ is a solution [satisfies all 3], so

$\boxed{\text{Yes, they intersect}}$.

at $(-2,-1,-7)$

$\langle -2,-2,-10\rangle \times \langle -9,9,30\rangle$

$= -2\cdot 3 \cdot \langle 1,1,5\rangle \times \langle -3,3,10\rangle = -6 \cdot \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 5 \\ -3 & 3 & 10 \end{vmatrix} = -6\langle -5,-25,6\rangle$

Let $\vec{n} = \langle 5, 25, -6\rangle$

$$\boxed{5(x+2) + 25(y+1) - 6(z+7) = 0}$$

3. Consider the lines $\ell(t) = (0,1,3) + t(2,1,5)$ [redacted] and $\ell_3(t) = (3,0,1) + t(-2,-1,1)$. Are they parallel, perpendicular, or neither? Do they intersect or not? If they intersect or are parallel, find an xyz-equation of the plane containing them. Otherwise find the distance between them.

$\boxed{\text{Perpendicular}}$: $(2,1,5)\cdot(-2,-1,1) = 0$

Intersect? $\begin{cases} 2t = 3-2s \\ 1+t = -s & ② \\ 3+5t = 1+s & ③ \end{cases}$ ②+③: $4+6t = 1$

$t = -\frac{1}{2} \overset{②}{\Rightarrow} s = -\frac{1}{2}$ is not a solution [fails ①], so

$\boxed{\text{they do not intersect}}$.

$\langle 2,1,5\rangle \times \langle -2,-1,1\rangle$

$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 5 \\ -2 & -1 & 1 \end{vmatrix} = \langle 6,-12,0\rangle$

$= 6\underbrace{\langle 1,-2,0\rangle}_{\vec{n}}$

$\vec{v} = (3,0,1)-(0,1,3)$
$= \langle 3,-1,-2\rangle$

distance $= \left|\text{proj}_{\vec{n}} \vec{v}\right| = \dfrac{|n\cdot v|}{\|v\|} = \dfrac{3+2+0}{\sqrt{9+1+4}} = \boxed{\dfrac{6}{\sqrt{14}}}$

4. Consider the two planes given by equations

$$3x - y + z = 4$$
$$2x + y - 2z = 6.$$

Find an equation of the line that is the intersection of these planes.

$$\vec{v} = \langle 3,-1,1\rangle \times \langle 2,1,-2\rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & 1 \\ 2 & 1 & -2 \end{vmatrix} = \langle 1, 8, 5\rangle$$

pt: $\begin{cases} 3x-y+z=4 \\ 2x+y-2z=6 \end{cases}$ Choose $z=0$; ①+②: $5x=10$
$x=2 \Rightarrow y=2$

$$\boxed{\ell(t) = (2,2,0) + t(1,8,5)}$$

5. Consider the two planes given by equations

$$3x + 12y - 3z = 1$$
$$2x + 8y - 2z = 7.$$

Find the distance between them.

$P = (\frac{1}{3}, 0, 0)$ on first plane

$Q = (\frac{7}{2}, 0, 0)$ on second plane

$\vec{PQ} = (\frac{21}{6}, 0, 0) - (\frac{2}{6}, 0, 0)$
$= \langle \frac{19}{6}, 0, 0\rangle$

$\vec{n} = \langle 1, 4, -1\rangle$ is normal to both planes

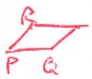
$$\text{distance} = \left|\text{proj}_{\vec{n}} \vec{PQ}\right| = \frac{\langle \frac{19}{6},0,0\rangle \cdot \langle 1,4,-1\rangle}{|\langle 1,4,-1\rangle|}$$

$$= \frac{19/6}{\sqrt{1+16+1}} = \boxed{\frac{19}{6\sqrt{18}}}$$

6. Consider the triangle with vertices $(1, 0, 5)$, $(2, -1, 2)$, and $(3, 1, -1)$.

P Q R

(a) Find its area.

$\text{Area} = \frac{1}{2}\text{Area}($ 

$)$

$\overrightarrow{PQ} = \langle 1, -1, -3 \rangle$

$\overrightarrow{PR} = \langle 2, 1, -6 \rangle$

$= \frac{1}{2} \| \overrightarrow{PQ} \times \overrightarrow{PR} \|$

$= \frac{1}{2} \| \langle 9, 0, 3 \rangle \| =$ [illegible] $= \frac{3}{2}\sqrt{10}$

(b) Is it acute, right, or obtuse?

$\overrightarrow{QR} = \langle 1, 2, -3 \rangle$

$\overrightarrow{QP} = \langle -1, 1, 3 \rangle$

$\overrightarrow{QR} \cdot \overrightarrow{QP} = -1 + 2 - 9 < 0$

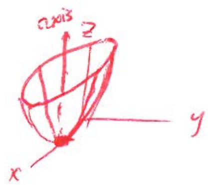
so the angle @ Q is obtuse,

hence the triangle is obtuse.

7. Identify by name the following surfaces, and sketch them.

(a) $z = (x-1)^2 + 4y^2$

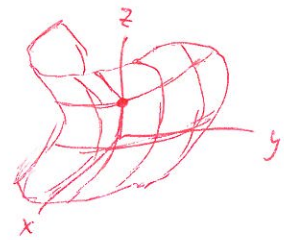
(elliptic) paraboloid



(b) $(z-1)^2 = x + 4y^2$

$x = (z-1)^2 - 4y^2$

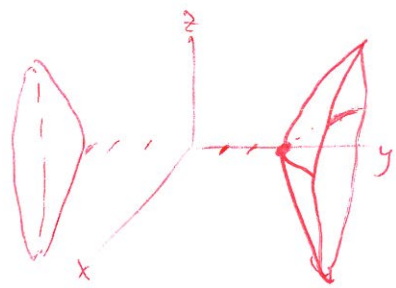
hyperbolic paraboloid



(c) $\frac{x^2}{4} - \frac{y^2}{9} + \frac{z^2}{25} = -1$

hyperboloid of two sheets

(when $y=0$, empty cross-section)

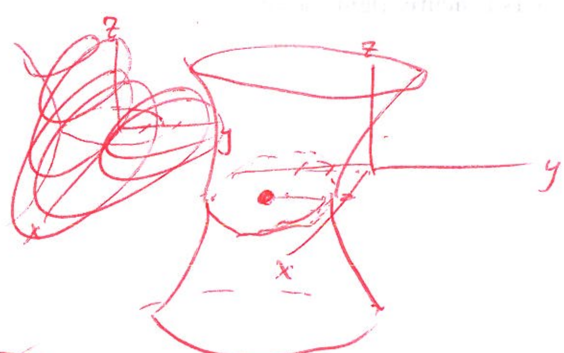


(d) $x^2 + y^2 - 2x + 4y = z^2 - 3$

$(x-1)^2 + (y+2)^2 = z^2 + 2$

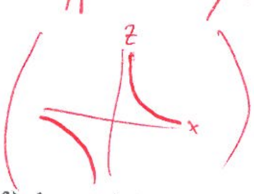
hyperboloid of one sheet

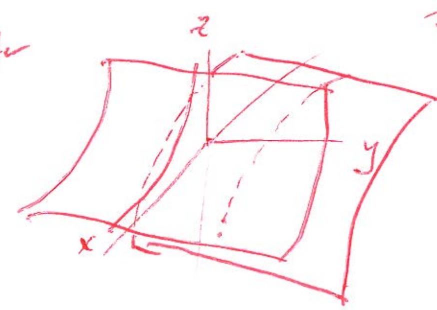
(when $z=0$, still a circle)



(e) $1 = xz$

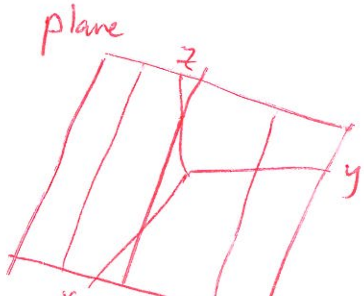
hyperbolic cylinder





(f) $1 = x + z$

plane



8. Suppose a particle moves in the plane, with position at time t given by $\underbrace{(t^2, t^3)}_{r(t)}$ at time t.

(a) Find the velocity at time $t = 9$.

$v = r' = \langle 2t, 3t^2 \rangle$

@ $t=9$, $\langle 18, 243 \rangle = 9\langle 2, 27 \rangle$

(b) Find the acceleration at time $t = 9$.

$a = v' = \langle 2, 6t \rangle$

@ $t=9$, $\langle 2, 54 \rangle = 2\langle 1, 27 \rangle$

(c) Find the tangential component of acceleration at time $t = 9$.

$$\vec{a}_T = \mathrm{proj}_{\vec{v}}\vec{a} = \frac{v \cdot a}{v \cdot v} v = \frac{36 + \cdots}{18^2 + 243^2} \langle 18, 243 \rangle$$

$$= \frac{9 \cdot 2 \cdot \langle 2, 27 \rangle \cdot \langle 1, 27 \rangle}{81 \cdot \langle 2, 27 \rangle \cdot \langle 2, 27 \rangle} \cdot 9\langle 2, 27 \rangle$$

$$= \frac{2(2 + 27^2)}{4 + 27^2} \langle 2, 27 \rangle$$

(d) Find the normal component of acceleration at time $t = 9$.

$\vec{a}_N = \vec{a} - \vec{a}_T$

(e) What do you know about how the speed of the particle is changing at $t = 9$?

increasing; [redacted]

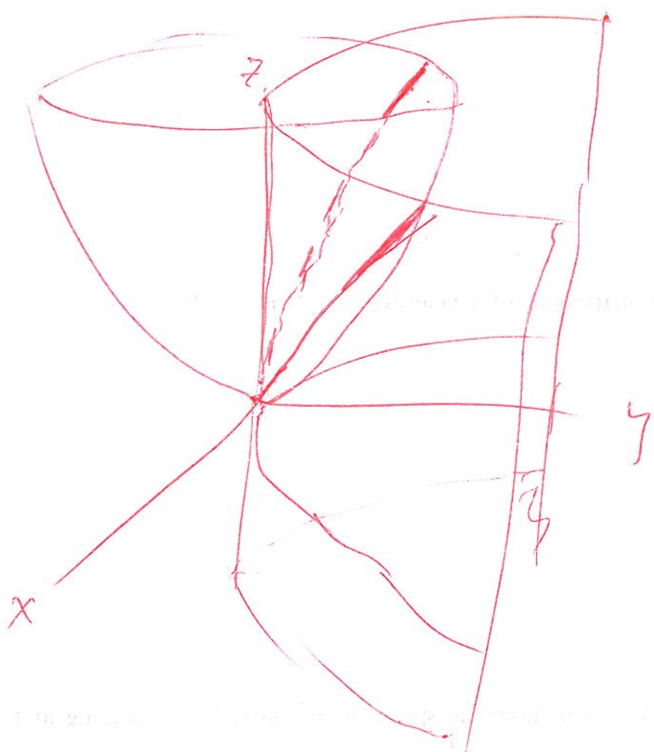
$\vec{a}_T$ is in the direction of $\vec{v}$.

9. Parametrize the intersection of the surfaces $z = 4x^2 + y^2$ and $y = x^2$.

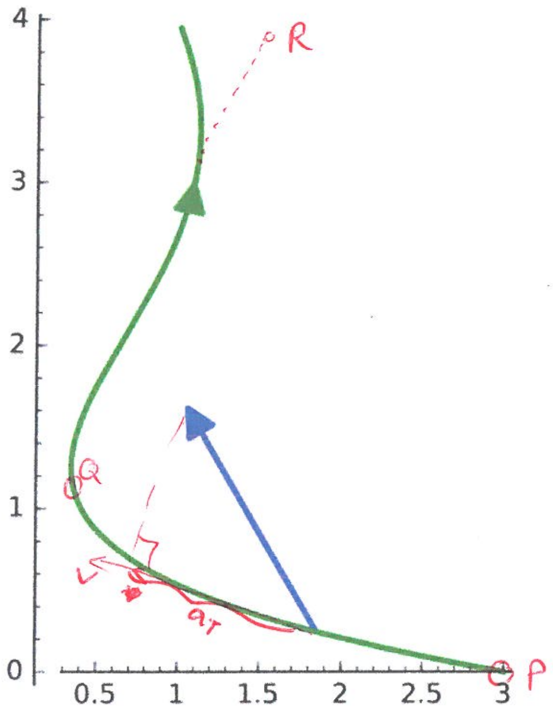
Let $x = t$; then $y = t^2$, &

$z = 4t^2 + (t^2)^2 = 4t^2 + t^4$

$\boxed{\langle t, t^2, 4t^2+t^4 \rangle}$



10. Here is a plot of a particle's position (in green). In blue is the particle's acceleration at a particular time (with its tail located at the particle's position at that time).



(a) Is the particle speeding up or slowing down at that time? Why?

Speeding up: $\vec{a}$ is generally in the same direction as $\vec{v}$, i.e. $a \cdot v > 0$, i.e. $\vec{a}_T$ is in the direction of $\vec{v}$.

(b) If you were given the position function $\mathsf{r}(t)$ and that t ran from 1 to 3, then give a formula for the distance traveled. Estimate the distance traveled.

distance traveled = arc length = $\int_1^3 |r'(t)|\,dt$

$\approx |PQ| + |QR| \approx |\langle -2.5, 1\rangle| + |\langle 1, 3\rangle|$

$= \boxed{\sqrt{\frac{25}{4}+1} + \sqrt{1+9}}$

11. Compute the curvature of a circle of radius r.

Put the circle's center at the origin. Then we can parametrize it as

$\vec{r}(t) = \langle r\cos t, r\sin t\rangle$, $t \in [0, 2\pi]$.

Need to treat as in $\mathbb{R}^3$ to take cross product

$$K = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3}$$

$\vec{r}\,'(t) = \langle -r\sin t, r\cos t\rangle$

$\vec{r}\,''(t) = \langle -r\cos t, -r\sin t\rangle$

$\vec{r}\,' \times \vec{r}\,'' = \langle 0, 0, r^2\sin^2 t + r^2\cos^2 t\rangle = \langle 0, 0, r^2\rangle$

$|\vec{r}\,' \times \vec{r}\,''| = r^2$

$|\vec{r}\,'| = \sqrt{r^2\sin^2 t + r^2\cos^2 t} = r$

$$\therefore K = \frac{r^2}{r^3} = \boxed{\frac{1}{r}}$$

12. Consider the vector function $\mathbf{r}(t) = \langle \cos t,\ \sin t,\ \ln(\cos t)\rangle$ ($t \in [-\pi/2, \pi/2]$). Find $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ at the point $(1, 0, 0)$. Find an equation for the tangent line to the curve at $(1, 0, 0)$.

$\vec{r} = (1,0,0)$ @ $t = 0$

$$\hat{T} = \frac{r'}{|r'|} = \frac{\left\langle -\sin t, \cos t, \frac{-\sin t}{\cos t}\right\rangle}{\sqrt{\sin^2 t + \cos^2 t + \frac{\sin^2 t}{\cos^2 t}}}$$

@ $t=0$: $\boxed{\langle 0, 1, 0\rangle}$

$$= \frac{\langle -\sin t, \cos t, -\tan t\rangle}{\sqrt{1+\tan^2 t}} = \frac{\langle -\sin t, \cos t, -\tan t\rangle}{\sqrt{\sec^2 t}} = \cos t\langle -\sin t, \cos t, -\tan t\rangle$$

(if $\cos t > 0$, & it is at $t=0$)

$= \langle -\sin t\cos t, \cos^2 t, -\sin t\rangle$

$$\hat{N} = \frac{T'}{|T'|} = \frac{\langle \sin^2 t - \cos^2 t, -2\sin t\cos t, -\cos t\rangle}{\sqrt{\sin^4 t - 2\sin^2 t\cos^2 t + \cos^4 t + 4\sin^2 t\cos^2 t + \cos^2 t}}$$

@ $t=0$: $\boxed{\frac{\langle -1, 0, -1\rangle}{\sqrt{2}}}$

$$\hat{B} = \hat{T} \times \hat{N} = \frac{1}{\sqrt{2}}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 0 \\ -1 & 0 & -1 \end{vmatrix}$$

$$\boxed{= \frac{1}{\sqrt{2}}\langle -1, 0, 1\rangle}$$

$$\boxed{\ell(t) = (1,0,0) + t\underbrace{(0,1,0)}_{\hat{T}}}$$

13. **Find the following limits if they exist.**

(a) $$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

DNE: Along $x=0$, function is 0

Along $y=x$, function is $\frac{x^2}{2x^2} = \frac{1}{2}$

(b) $$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} = 0$$

Polar coordinates: $\lim_{r\to 0^+} \frac{r^3\cos^2\theta\sin\theta}{r^2} = \lim_{r\to 0^+} r\underbrace{\cos^2\theta\sin\theta}_{\text{bounded}} = 0$

(c) $$\lim_{(x,y)\to(0,0)} \frac{x^2ye^{-1/y^2}}{x^2+y^2} = 0$$

$e^{-1/y^2} \to 0$ as $y\to 0$,

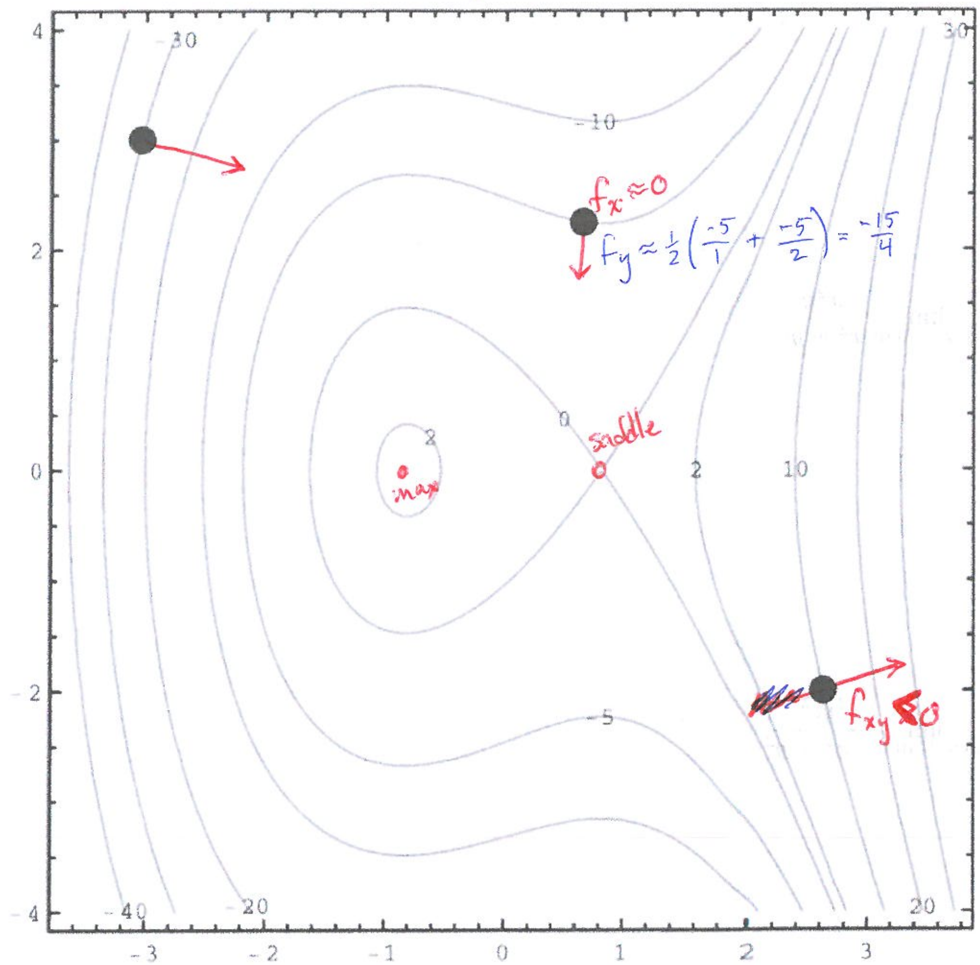
& $\frac{x^2y}{x^2+y^2} \to 0$ by (b), so the product also $\to 0$.

(d) $$\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^6+y^2}$$

DNE: Along $x=0$, function is 0

Along $y=x^3$; ---- $\frac{x^3(x^3)}{x^6+(x^3)^2} = \frac{x^6}{2x^6} = \frac{1}{2}$

14. Below is a plot of several level curves of a function $f(x, y)$ (they are NOT at equally-spaced heights). At the indicated points, sketch in the gradient vectors. At the lower-right point, is f_{xy} positive, negative, or zero? At the upper-central point, estimate f_x. Find the (approximate) locations of the critical points of f, then classify them. & f_y



15. Find the linearization of $f(x, y) = x^2e^y$ based at the point $(1, 0)$.

$f_x = 2xe^y$ @ $(1,0)$: 2

$f_y = x^2e^y$ 1

$L(x,y) = f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0)$

$\boxed{L(x,y) = 1 + 2(x-1) + 1(y)}$

16. Approximate $f(1.1, 0.9)$ using the linearization above.

$f(1.1, 0.9) \approx L(1.1, 0.9) = 1 + 2(0.1) + 1(0.9)$

$= \boxed{2.1}$

17. Suppose $f(x, y, z)$, $x(s, t)$, $y(s, t)$, and $z(s, t)$ are all differentiable everywhere. You are given the following:

$x(1,2) = 3$	$y(1,2) = 4$	$z(1,2) = 5$
$x_t(1,2) = -1$	$y_t(1,2) = -2$	$z_t(1,2) = -3$
$x_s(1,2) = 7$	$y_s(1,2) = 8$	$z_s(1,2) = 9$
$f(-1,-2,-3) = 11$	$f(0,0,0) = 12$	$f(3,4,5) = 10$
$f_x(-1,-2,-3) = -7$	$f_x(0,0,0) = 0$	$f_x(3,4,5) = -4$
$f_y(-1,-2,-3) = -8$	$f_y(0,0,0) = -9$	$f_y(3,4,5) = -5$
$f_z(-1,-2,-3) = -10$	$f_z(0,0,0) = -11$	$f_z(3,4,5) = -6$

Compute $\frac{\partial f}{\partial t}$ at $(s, t) = (1, 2)$.

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

At $(s,t) = (1,2)$, $(x,y,z) = (3,4,5)$

$$= -4(-1) - 5(-2) - 6(-3)$$

$$= 4 + 10 + 18$$

$$= \boxed{32}$$

18. Find and classify the local extrema of $f(x, y) = x^3 + y^3 - 3x^2 - 3y^2 - 9x$.

$$\left.\begin{array}{l} f_x = 3x^2 - 6x - 9 = 0 \\ f_y = 3y^2 - 6y = 0 \end{array}\right\} \quad \begin{array}{l} 3(x-3)(x+1) = 0 \Rightarrow x = -1 \text{ OR } x = 3 \\ 3y(y-2) = 0 \Rightarrow y = 0 \text{ OR } y = 2 \end{array}$$

Four critical points: $(-1,0)$, $(-1,2)$, $(3,0)$, $(3,2)$

$$D = \begin{vmatrix} 6x-6 & 0 \\ 0 & 6y-6 \end{vmatrix} = 36(x-1)(y-1)$$

$D(-1,0) > 0$ $\quad f_{xx}(-1,0) < 0 \Rightarrow$ local max @ $(-1,0)$

$D(3,2) > 0$ $\quad f_{xx}(3,2) > 0 \Rightarrow$ local min @ $(3,2)$

$\left.\begin{array}{l} D(-1,2) < 0 \\ D(3,0) < 0 \end{array}\right\}$ saddle points @ $(-1,2)$ & $(3,0)$

19. (a) Find the maximum and minimum values of $f(x, y) = xy$ on the disk $x^2 + y^2 \leq 4$.

Interior: $\left.\begin{array}{l} f_x = y = 0 \\ f_y = x = 0 \end{array}\right\} \Rightarrow (0,0)$

Boundary: Lagrange: $\left\{\begin{array}{l} y = \lambda 2x \\ x = \lambda 2y \\ x^2+y^2=4 \end{array}\right.$ $\xrightarrow{\text{substitute}}$ $x = 2(2x\lambda)\lambda = 4x\lambda^2$

$g(x,y) = x^2+y^2 = 4$

$\Rightarrow x = 0$ OR $4\lambda^2 = 1$

$\Downarrow$ $\qquad \lambda = \pm\frac{1}{2}$

$y = 0$ $\qquad \overset{(1)}{\Rightarrow} y = \pm x$

↯ $\qquad \overset{(3)}{\Rightarrow} x^2 + (\pm x)^2 = 4$

$2x^2 = 4$

$x = \pm\sqrt{2}$

$y = \pm\sqrt{2}$

$f(0,0) = 0$

$f(-\sqrt{2}, -\sqrt{2}) = f(\sqrt{2}, \sqrt{2}) = 2$ Max

$f(-\sqrt{2}, \sqrt{2}) = f(\sqrt{2}, -\sqrt{2}) = -2$ min

(b) Draw the disk together with any critical points you found and the level curves of f corresponding to the values at those critical points.