Name: ______________________________

- **READ THE FOLLOWING DIRECTIONS!**
- **Do NOT open the exam until instructed to do so.**
- You have 120 minutes to complete this exam. When you are told to stop writing, do it or you will lose all points on the page you write on.
- You may not communicate with other students during this test.
- No written materials of any kind are allowed. No scratch paper is allowed except as given by the proctors.
- No phones, calculators, or any other electronic devices are allowed for any reason, including checking the time (a simple wristwatch is fine).
- Any case of cheating will be taken extremely seriously.
- Show all your work and explain your answers.
- Before turning in your exam, check to make certain you've answered all the questions.

I will give you the following formulas on the final exam. You are responsible for knowing their names, what they mean, and how to use them.

$$\begin{aligned}\iint_D (Q_x - P_y)\,dA &= \oint_C P\,dx + Q\,dy\\ \iint_\Sigma \operatorname{curl}\vec{F}\cdot d\vec{S} &= \int_C \vec{F}\cdot d\vec{r}\\ \iiint_E \operatorname{div}\vec{F}\,dV &= \iint_\Sigma \vec{F}\cdot d\vec{S}\end{aligned}$$

$$\begin{aligned}\cos^2 t &= \frac{1}{2}(1+\cos(2t))\\ \sin^2 t &= \frac{1}{2}(1-\cos(2t))\end{aligned}$$

1. (a) Suppose there is a curve C in $\mathbb{R}^3$ that is the boundary of some surface R, and that the vector field $\vec{F}$ has the peculiar property that $\operatorname{curl}\vec{F}$ is a unit normal vector for R at every point. Find $\int_C \vec{F} \bullet \vec{dr}$ in terms of some geometric measurement(s). (Don't worry about the direction.)

   (b) What if $\operatorname{curl}\vec{F}$ is instead a unit tangent vector to R at every point?

2. Consider the vector field $\vec{F}(x, y, z) = \langle 1 - xz + \frac{1}{2}z^2, 1 - yz, xz \rangle$. Find the work done by $\vec{F}$ on a particle moving counterclockwise along the unit circle in the xy-plane by...

   (a) ...directly computing a path integral.

   (b) ...using Green's Theorem on the 2D field $\vec{F_0} = \langle y, 1-x \rangle$ in the xy-plane. (This field is the restriction of $\vec{F}$ to the xy-plane.)

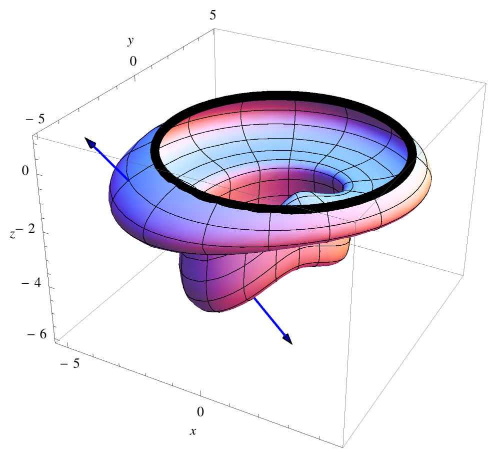
   (c) ...using Stokes's Theorem and the disk in the xy-plane (with $z = 0$).

   (d) ...using Stokes's Theorem and the hemisphere $z = \sqrt{1 - x^2 - y^2}$.

   (e) ...using Stokes's Theorem and the paraboloid $z = 1 - x^2 - y^2$.

   (Check out problem 1 and how it relates here.)

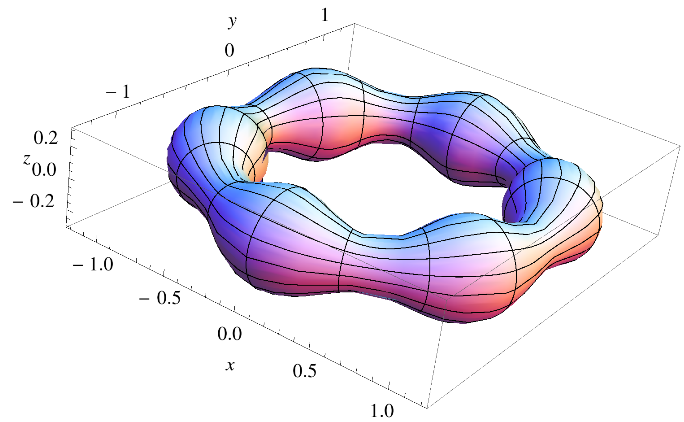
3. The surface R is shown below; its boundary is the circle of radius 2 in the xy-plane. Let $\vec{F}(x, y, z) = \langle x, y + z^2, y - z \rangle$.



(a) Find the direction of the flux of $\vec{F}$ across R. (Is it in the direction of the displayed normal vectors or opposite?)

(b) Find the flux (amount and direction) of $\text{curl}\,\vec{F}$ across R.

4. The surface S is shown below; it has no boundary. Let $\vec{F}(x, y, z) = \langle x, y + z^2, y - z \rangle$.



(a) Find the direction of the flux of $\vec{F}$ across S. (Is it inward or outward?)

(b) Find the flux (amount and direction) of $\operatorname{curl} \vec{F}$ across S.

5. Let $\vec{F}(x, y, z) = \langle x^2 e^y, 2e^y - y^2 z, yz^2 \rangle$, and let R be the sphere of radius 2 centered at $(4, 2, 1)$. Is the flux of $\vec{F}$ across R inward or outward? (Note that you are not asked to find the quantity, just the direction.)

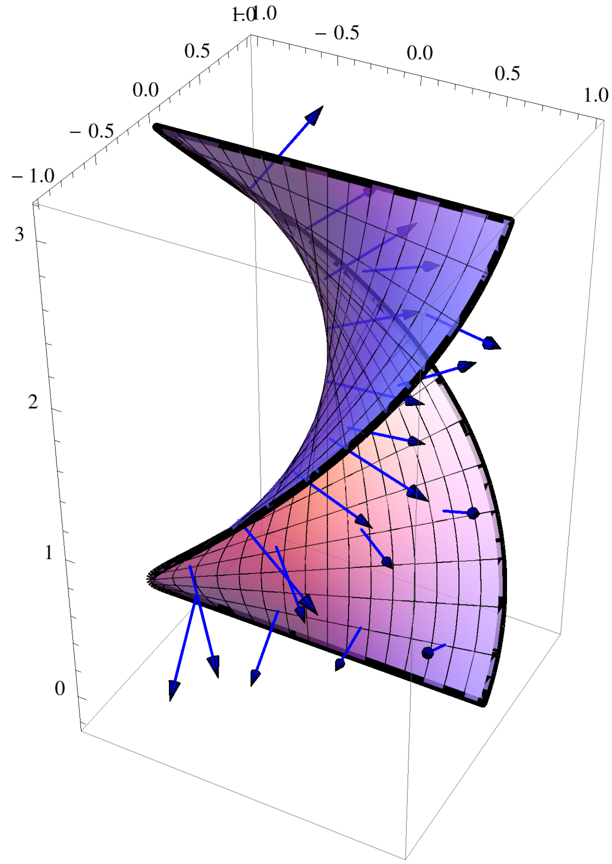
6. Consider the vector field $\vec{F}(x, y, z) = \langle 3x^2y, x^3 + e^z, ye^z \rangle$.

   (a) Verify that $\vec{F}$ is a gradient field. (Show enough work for me to know that you know what you're doing.)

   (b) What is the work done by $\vec{F}$ on a particle moving along the curve $\left\langle 27 + \cos t, 43 - \sin t, e^{84t^2} \sin t \right\rangle$, $t \in [0, 2\pi]$?

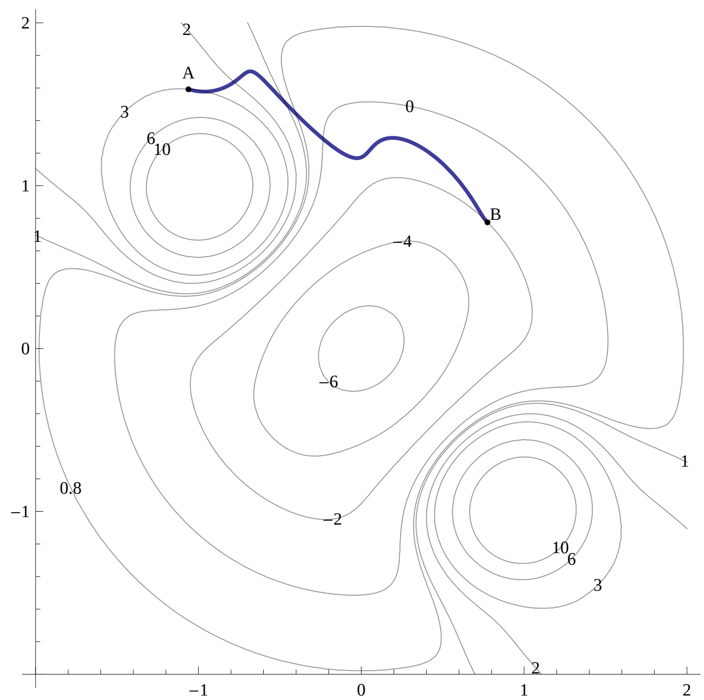
7. Use Stokes's Theorem to compute the work done by the force $\vec{F}(x, y, z) = \langle 2y, 3z, x \rangle$ in moving a particle along the triangle C with vertices $(0, 0, 0)$, $(1, 1, 1)$, and $(3, 0, 2)$ in that order.

8. Below is a surface with a selection of normal direction. Indicate on the picture the orientation of the boundary of the surface that "matches" in the sense of Stokes's Theorem. (To be clear: the normals shown point toward you near the bottom of the picture, and away from you near the top of the picture.)



9. Let Σ be the surface with parametrization $\langle u^2, uv, v\rangle$, with $u \in [-1, 1]$ and $v \in [0, 1]$. Suppose a sheet of metal in the shape of Σ has density xy^2 at each point. Set up a double integral in u, v that measures the mass of this sheet. (The integral should be over a region in the uv-plane, with no vectors involved.)

10. Below is shown a contour map of a function $f(x, y)$ on the rectangle D. Let $\vec{F}(x, y) = \nabla f(x, y)$. Briefly explain all your answers. (This question continues on the next page.)



(a) Which of the following is the best estimate of $\iint_D f\,dx\,dy$:

$-40 \qquad -20 \qquad 0 \qquad 20 \qquad 40$

(b) Find $\int_C \vec{F} \cdot d\vec{r}$, where C is the path going from A to B, as shown.

(c) Find and classify the critical points of f.

(d) Which of the following is $\vec{F}$?