Name: ____________________

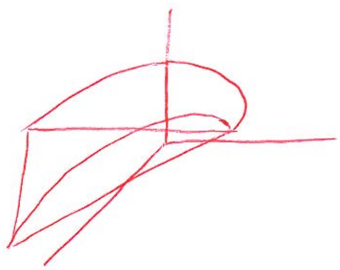
- **READ THE FOLLOWING DIRECTIONS!**
- **Do NOT open the exam until instructed to do so.**
- You have 75 minutes to complete this exam. When you are told to stop writing, do it or you will lose all points on the page you write on.
- You may not communicate with other students during this test.
- No written materials of any kind are allowed. No scratch paper is allowed except as given by the proctors.
- No phones, calculators, or any other electronic devices are allowed for any reason, including checking the time (a simple wristwatch is fine).
- Any case of cheating will be taken extremely seriously.
- Show all your work and explain your answers.
- Before turning in your exam, check to make certain you've answered all the questions.

You may or may not need the following formulas:

$$\sin^2 t = \frac{1}{2}(1 - \cos(2t))$$
$$\cos^2 t = \frac{1}{2}(1 + \cos(2t))$$

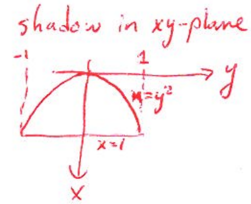
Question:	1	2	3	4	5	6	7	8	Total
Points:	10	6	8	2	7	10	6	3	52
Score:									

1. (10 points) Find the volume of the region bounded by $z = y$, $x = y^2$, $x = 1$, and $z = 1$.



$$\text{Volume} = \iiint_E 1\, dV$$

$$= \int_{-1}^{1} \int_{y^2}^{1} \int_{y}^{1} dz\, dx\, dy$$



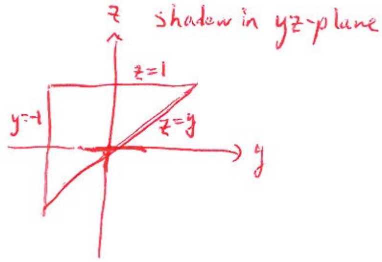
$$= \int_{-1}^{1} \int_{y^2}^{1} (1-y)\, dx\, dy = \int_{-1}^{1} (1-y)(1-y^2)\, dy = \int_{-1}^{1} (1-y-y^2+y^3)\, dy$$

$$= 2 - 0 - \frac{2}{3} + 0 = \frac{4}{3}$$

Also valid:

$$\int_{0}^{1} \int_{-\sqrt{x}}^{\sqrt{x}} \int_{y}^{1} dz\, dy\, dx$$

$$\int_{-1}^{1} \int_{-1}^{z} \int_{y^2}^{1} dx\, dy\, dz \qquad \int_{-1}^{1} \int_{y}^{1} \int_{y^2}^{1} dx\, dz\, dy$$



(dy first splits into two pieces, depending on whether the upper bound on y is $\sqrt{x}$ or z)

2. A metal sheet occupies the rectangle $[0, 7] \times [0, 9]$. It has non-uniform density; you are given the densities at various points:

R

	9	1	2	4	6	5	3	1	2
	8	0	1	3	4	5	3	2	1
	7	5	3	5	7	6	4	2	1
	6	2	4	5	6	7	4	3	2
	5	3	5	6	8	9	6	5	3
y	4	1	4	6	7	9	6	4	3
	3	2	3	4	6	7	5	4	3
	2	4	5	3	4	5	3	2	1
	1	5	6	4	3	3	1	1	2
	0	6	8	7	4	2	1	2	3
		0	1	2	3	4	5	6	7
$\rho(x, y)$					x				

(a) (3 points) Estimate the mass of the part of the sheet R with $1 \leq x \leq 5$ and $4 \leq y \leq 8$, using a Riemann sum with two subintervals in each direction (the book/WebAssign would say $m = n = 2$).

Using the Midpoint Rule (other methods are possible),

$$\text{mass} = \iint_R \rho \, dA \approx \Delta A \left(\rho(2,5) + \rho(2,7) + \rho(4,5) + \rho(4,7) \right)$$
$$= 4(6 + 5 + 9 + 6)$$
$$= 104$$

(b) (3 points) Estimate $\displaystyle\iint_R x\rho(x, y) \, dA$, using again a Riemann sum with two subintervals in each direction.

$$\approx 4\left(2\cdot\rho(2,5) + 2\cdot\rho(2,7) + 4\cdot\rho(4,5) + 4\cdot\rho(4,7) \right)$$
$$= 4(12 + 10 + 36 + 24)$$
$$= 328$$

[Note that the x-coordinate of the center of mass, $\bar{x}$, is given by $\frac{1}{m}\iint_R x\rho \, dA \approx \frac{328}{104} = 3.1538$, which is reasonable.]

3. (8 points) Set up the integral in spherical coordinates (*you do not have to evaluate it*):

$$\int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \int_0^{\sqrt{9-x^2-y^2}} \sqrt{x^2+y^2}\, dz\, dx\, dy$$

$\sqrt{x^2+y^2} = \sqrt{(\rho\sin\varphi\cos\theta)^2+(\rho\sin\varphi\sin\theta)^2} = \rho\sin\varphi$

$0 \le z \le \sqrt{9-x^2-y^2}$ is the top half of a ball of radius 3.

$\left.\begin{array}{l} -\sqrt{9-y^2} \le x \le \sqrt{9-y^2} \\ 0 \le y \le 3 \end{array}\right\}$ is the top half of the disk of radius 3.

$$\int_0^{\pi} \int_0^{\pi/2} \int_0^{3} (\rho\sin\varphi)(\rho^2\sin\varphi)\, d\rho\, d\varphi\, d\theta$$

4. (2 points) Circle 'True' or 'False' (1 point each):

(a) True False For any two functions f, g of three variables and any region E,
$\iiint_E (f+g)\, dV = \iiint_E f\, dV + \iiint_E g\, dV.$

(b) True False For any two functions $f(x,y)$ and $g(z)$ and any region E, if I is the shadow of E on the z-axis and D the shadow in the xy-plane, then
$\iiint_E f(x,y)g(z)\, dV = \left(\iint_D f(x,y)\, dA\right)\left(\int_I g(z)\, dz\right)$

5. (7 points) Evaluate $\iint_R \frac{3x-y}{x+3y}\,dA$, where R is the parallelogram enclosed by the lines $3x-y=2$, $3x-y=4$, $x+3y=5$, and $x+3y=7$.

Let $u = 3x-y$
$v = x+3y$

$\frac{1}{J} = \begin{vmatrix} 3 & -1 \\ 1 & 3 \end{vmatrix} = 10$

$$\int_5^7 \int_2^4 \frac{u}{v} \cdot \frac{1}{10}\,du\,dv$$

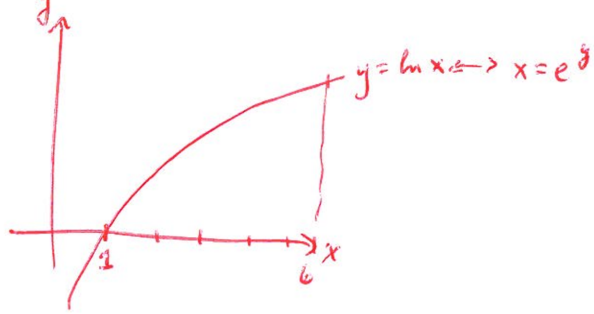
$$= \frac{1}{20} \int_5^7 \frac{1}{v}(16-4)\,dv$$

$$= \frac{12}{20} \ln v \Big|_5^7$$

$$= \frac{3}{5}(\ln 7 - \ln 5)$$

$$= \frac{3}{5} \ln \frac{7}{5}$$

6. (10 points) Compute $\int_1^6 \int_0^{\ln x} y\,dy\,dx$ by changing the order of integration.



$$= \int_0^{\ln 6} \int_{e^y}^{6} y\,dx\,dy$$

$$= \int_0^{\ln 6} 6y - ye^y\,dy$$

$$= \int_0^{\ln 6} 6y\,dy - \int_0^{\ln 6} ye^y\,dy$$

$u = y \quad dv = e^y dy$
$du = dy \quad v = e^y$

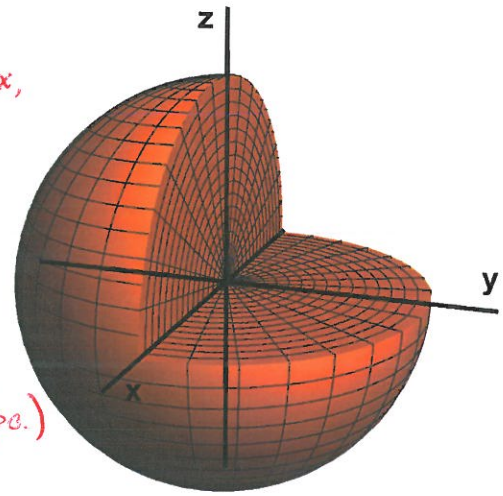
$$= 3(\ln 6)^2 - \left[ye^y - \int e^y dy \right]_0^{\ln 6}$$

$$= 3(\ln 6)^2 - 6\ln 6 + e^y \Big|_0^{\ln 6}$$

$$= 3(\ln 6)^2 - 6\ln 6 + 6 - 1$$

$$= 3(\ln 6)^2 - 6\ln 6 + 5$$

7. (6 points) Below is a region E. For each part, circle the sign of the integral and give brief justification:



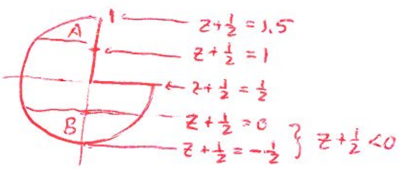
$\iiint_E xy\, dV$ $\quad + \quad 0 \quad -$

odd function of x, E is symmetric front-to-back

$\iiint_E y\, dV$ $\quad + \quad 0 \quad -$

more of E has $y<0$. (The whole ball would give $\iiint y\,dV=0$, but we cut out a chunk where $y>0$.)

$\iiint_E \left(z+\frac{1}{2}\right) dV$ $\quad + \quad 0 \quad -$



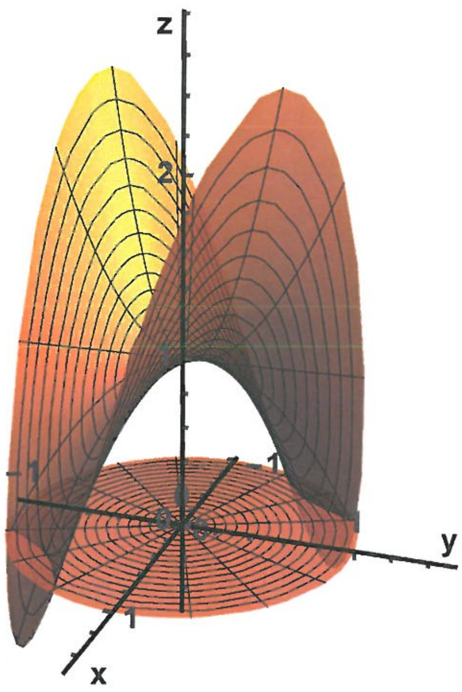
$z+\frac{1}{2}$ is mostly positive on E.

More concretely, the negative contribution is from B,

and $\iiint_B z+\frac{1}{2}\,dV \geq \iiint_B -\frac{1}{2}\,dV = -\frac{1}{2}\cdot \mathrm{Vol}(B)$, but $\iiint_A z+\frac{1}{2}\,dV \geq \iiint_A 1\,dV = \mathrm{Vol}(A) = \frac{1}{2}\mathrm{Vol}(B)$,

So the sum is ≥ 0

8. (3 points) Below is shown the graph of a function f over the unit disk R in the xy-plane. Estimate the average value of f on R.



It is 1, because of the symmetry (the two "crests" where $f>1$ have the same volume as the "dips" where $f<1$).

Scratch Paper - Do Not Remove