Name: ______________________________

- **READ THE FOLLOWING DIRECTIONS!**
- **Do NOT open the exam until instructed to do so.**
- You have 75 minutes to complete this exam. When you are told to stop writing, do it or you will lose all points on the page you write on.
- You may not communicate with other students during this test.
- No written materials of any kind are allowed. No scratch paper is allowed except as given by the proctors.
- No phones, calculators, or any other electronic devices are allowed for any reason, including checking the time (a simple wristwatch is fine).
- Any case of cheating will be taken extremely seriously.
- Show all your work and explain your answers.
- Before turning in your exam, check to make certain you've answered all the questions.

Question:	1	2	3	4	5	6	7	8	9	Total
Points:	8	6	9	5	9	9	16	3	0	65
Score:										

1. Suppose f is a function with continuous second partial derivatives. You are given the following information about f and its derivatives at five points:

	(0,0)	(1,1)	(2,3)	(1,5)	(−1,1)
f	3	0	−1	2	5
f_x	0	0	0	0	0
f_y	0	−1	0	0	0
f_{xx}	1	2	−2	2	−1
f_{xy}	3	−1	1	−1	3
f_{yy}	8	−1	−2	2	2
D	-1	-3	3	3	-11

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

(a) (5 points) Classify each of the given points as a local maximum, local minimum, saddle point, or none of those.

2nd Deriv Test:
If $\nabla f(P) = 0$ &
- $D(P) > 0$ & $f_{xx}(P) > 0$, then P is local min
- $D(P) > 0$ & $f_{xx}(P) < 0$, then P is local max
- $D(P) < 0$, then P is saddle point

(0,0) saddle

(1,1) none (not a critical point: $f_y \neq 0$)

(2,3) max

(1,5) min

(−1,1) Saddle

(b) (3 points) Estimate $f(0.9, 1.1)$ using the linearization of f (at the most appropriate base point).

↳ (1,1) is closest given point to (0.9, 1.1)

$$L(x,y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$$
$$= 0 + 0 - 1(y-1)$$
$$= -y+1$$

$$f(0.9, 1.1) \approx L(0.9, 1.1) = -1.1 + 1 = \boxed{-0.1}$$

2. Find the following limits if they exist. Fully justify your answers.

(a) (3 points) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4+y^4}{x^2+y^2}$

Polar: $\displaystyle = \lim_{r\to 0^+} \frac{r^4\cos^4\theta + r^4\sin^4\theta}{r^2} = \lim_{r\to 0^+} r^2 \underbrace{(\cos^4\theta+\sin^4\theta)}_{\text{bounded}} = \boxed{0}$

(b) (3 points) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^3-yx^3}{x^4+y^4}$

Along x-axis, $\displaystyle\lim_{(x,0)\to(0,0)} \frac{0}{x^4} = 0.$

Along $y=2x$, $\displaystyle\lim_{(x,2x)\to(0,0)} \frac{x(2x)^3-(2x)x^3}{x^4+(2x)^4} = \lim_{x\to 0} \frac{8x^4-2x^4}{x^4+16x^4} = \frac{6}{17}.$

So the limit in question $\boxed{\text{DNE}}$.

3. The lines $\ell_1(t) = (3,1,4) + t(-1,-1,1)$ and $\ell_2(t) = (7,0,8) + t(6,1,2)$ **do intersect**.

(a) (3 points) Find the point where they intersect.

$$\left\{\begin{array}{l} 3 - t = 7 + 6s \\ 1 - t = s \\ 4 + t = 8 + 2s \end{array}\right.$$

$\oplus$ $5 = 8 + 3s$

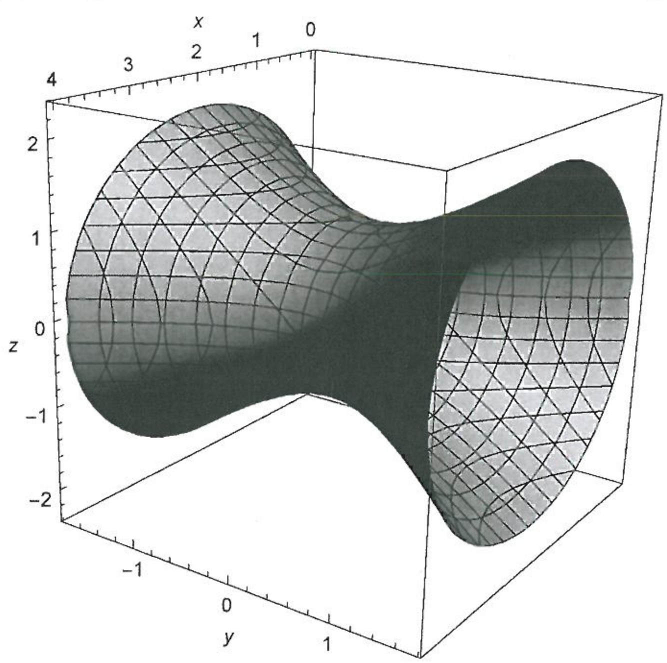
$-1 = s \Rightarrow \boxed{(1, -1, 6)}$

(b) (6 points) Find an equation of the plane containing both lines.

$$\vec{n} = \langle -1, -1, 1\rangle \times \langle 6, 1, 2\rangle = \langle -3, 8, 5\rangle$$

$$\boxed{-3(x-1) + 8(y+1) + 5(z-6) = 0}$$

4. (5 points) Match the given surface to its equation. Also give the name of the surface.



A. $x^2 + y^2 - z^2 = 1$

B. $x^2 + y^2 - z^2 = -1$

C. $x^2 - y^2 + z^2 = -1$

D. $x^2 - y^2 + z^2 = 1$

E. $(x-2)^2 - y^2 + z^2 = -1$

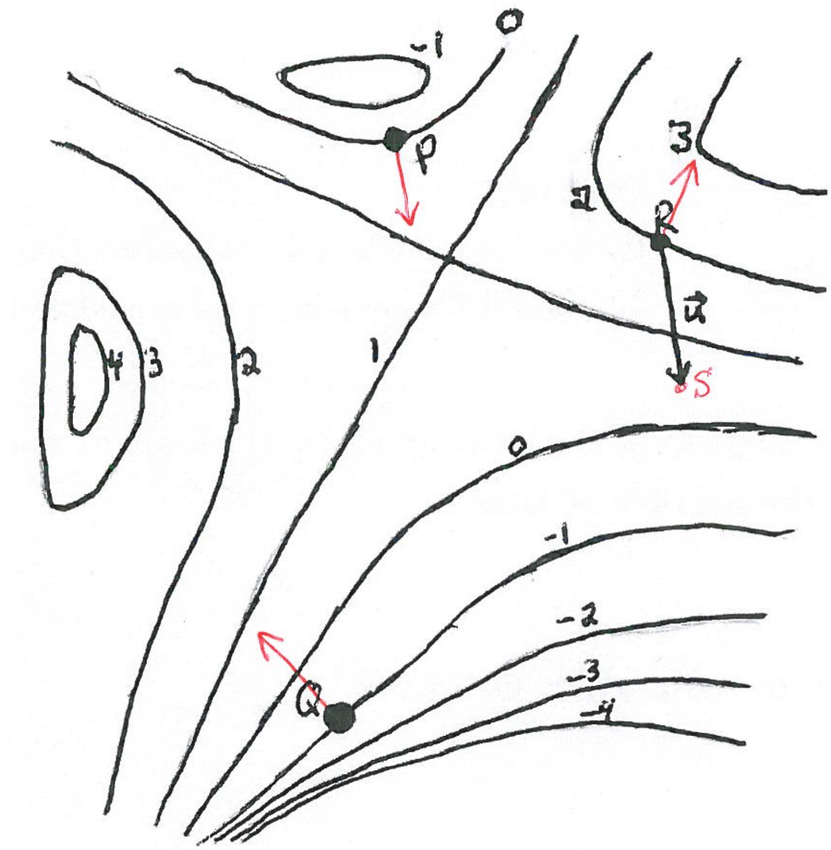
F. $(x-2)^2 + y^2 - z^2 = 1$

G. $(x-2)^2 - y^2 + z^2 = 1$

H. $(x-2)^2 + y^2 - z^2 = -1$

Name of surface: hyperboloid of one sheet

5. Below is a contour plot of a function f, together with several points and a unit vector $\mathbf{u}$.



(a) (3 points) Sketch in ∇f at each labelled point.

(b) (2 points) Estimate $D_{\mathbf{u}}f(R)$.

$$\approx \frac{\Delta f}{|u|} \approx \frac{\overset{f(S)}{0.5} - \overset{f(R)}{2}}{1} = -1.5$$

(c) (4 points) Circle below the sign of each second partial derivative of f at Q.

$f_{xx}(Q)$: + − 0 Level curves are evenly spaced along horizontal line through Q.

$f_{xy}(Q)$: + − 0 Moving north through Q, level curves have more horizontal spacing, i.e. f_x is becoming less negative, i.e. f_x is increasing, i.e. $(f_x)_y > 0$.

$f_{yy}(Q)$: + − 0 Similar to

6. Consider the function $f(x, y) = x^2 + 4y^2$, and the disk D given by $x^2 + y^2 \leq 1$.

   (a) (2 points) Find f_x and f_y.

   $f_x = 2x \quad f_y = 8y$

   (b) (5 points) Find the maximum and minimum values of f on D.

   Interior: $\begin{cases} 2x = 0 \\ 8y = 0 \end{cases} \Rightarrow (x, y) = (0, 0).$

   Boundary: Lagrange

   $\begin{cases} 2x = \lambda(2x) \Rightarrow \lambda = 1 \text{ OR } x = 0 \\ 8y = \lambda(2y) \\ x^2 + y^2 = 1 \end{cases}$

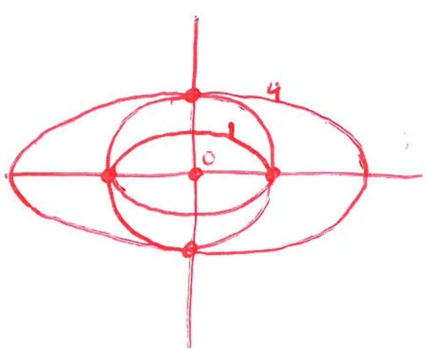
   $\lambda = 1 \Rightarrow y = 0 \Rightarrow x = \pm 1$; $x = 0 \Rightarrow y = \pm 1$

   $f(0,0) = \boxed{0 \text{ Min}}$

   $f(\pm 1, 0) = 1$

   $f(0, \pm 1) = \boxed{4 \text{ Max}}$

   (c) (2 points) Draw D together with any critical points you found in (b) and the level curves of f corresponding to the values at those critical points.



7. (16 points) Circle 'True' or 'False' (2 points each, no partial credit):

(a) True False For any two vectors $\mathbf{u}$ and $\mathbf{v}$, we have $|\mathbf{u} \times \mathbf{v}| = |\mathbf{v} \times \mathbf{u}|$.

$u \times v = -v \times u$, & these have the same magnitude

(b) True False For any three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, we have $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$.

$u = v = \hat{\imath},\ w = \hat{\jmath} \Rightarrow$ Left side $= \hat{\imath} \times (\hat{\imath} \times \hat{\jmath}) = \hat{\imath} \times \hat{k} = -\hat{\jmath}$

Right side $= (\hat{\imath} \times \hat{\imath}) \times \hat{\jmath} = \vec{0} \times \hat{\jmath} = \vec{0}$

(c) True False For any vector $\mathbf{v}$, we have $|\mathbf{v} \cdot \mathbf{v}| = |\mathbf{v}|^2$.

$v \cdot v = |v|^2$; adding absolute values doesn't hurt since it is already ~~positive~~ nonnegative

(d) True False For any vector $\mathbf{v}$, we have $|\mathbf{v} \times \mathbf{v}| = |\mathbf{v}|^2$.

Always $v \times v = \vec{0}$

(e) True False For any vector function $\mathbf{r}(t)$, $\left|\int_a^b \mathbf{r}'(t)\,dt\right| = \int_a^b |\mathbf{r}'(t)|\,dt$.

net displacement distance traveled

(f) True False If a curve in the plane is concave down, then its curvature is negative.

$\kappa = \dfrac{|r' \times r''|}{|r'|^3}$ is always non-negative

(g) True False If $\mathbf{r}(t)$ is the position of a particle at time t, then it must be true that $\text{proj}_{\mathbf{B}}\, \mathbf{r}'' = \mathbf{0}$ at every value of t (provided $\mathbf{B} \neq \mathbf{0}$).

$\text{proj}_B\, a$; acceleration only has components in the tangential & principal normal directions

(h) True False If $\mathbf{r}(t)$ is the position of a particle at time t, and $\mathbf{r}' \cdot \mathbf{r}'' < 0$ at time t, then $\frac{d}{dt}|\mathbf{r}'| < 0$ at time t.

$v \cdot a$

speed

particle is slowing down

8. (3 points) Compute $\text{proj}_{\langle 1,2,2\rangle}\langle 3,-1,5\rangle$.

$$= \frac{\langle 1,2,2\rangle \cdot \langle 3,-1,5\rangle}{\langle 1,2,2\rangle \cdot \langle 1,2,2\rangle} \langle 1,2,2\rangle$$

$$= \boxed{\frac{11}{9}\langle 1,2,2\rangle}$$

9. (6 points (bonus)) Compute the arc length of the curve $\mathbf{r}(t) = \langle \cos t, \sin t, \ln \cos t \rangle$, $t \in [0, \pi/4]$.

$$\text{Arc length} = \int_0^{\pi/4} |r'(t)|\, dt$$

$$r'(t) = \left\langle -\sin t, \cos t, -\frac{\sin t}{\cos t} \right\rangle = \langle -\sin t, \cos t, -\tan t \rangle$$

$$|r'(t)| = \sqrt{\sin^2 t + \cos^2 t + \tan^2 t}$$

$$= \sqrt{1 + \tan^2 t}$$

$$= \sqrt{\sec^2 t} = |\sec t| = \sec t$$

$t \in [0, \pi/4] \Rightarrow \sec t > 0$

$$\int_0^{\pi/4} \sec t\, dt = \ln|\sec t + \tan t| \Big]_0^{\pi/4}$$

$$= \ln|\sqrt{2} + 1| - \ln|1 + 0|$$

$$= \boxed{\ln(\sqrt{2} + 1)}$$