Research Statement

Benjamin Reiniger

October 12, 2015

1 Introduction

My research lies in combinatorics, which can be described as the study of finite (or perhaps countably infinite) structures. In particular, I work with graphs and hypergraphs–which represent symmetric relationships between objects, such as connections in phone and internet networks–and partially ordered sets ("posets")–which represent asymmetric but transitive relationships between objects, such as people's preferences.

Within these fields, I am particularly interested in coloring, structural and extremal problems, poset dimension, and competitive optimization. I am especially interested in the interaction of combinatorics with other areas of mathematics, and have some background in topology and geometric group theory.

2 Graphs and hypergraphs

2.1 Colorings

A *proper coloring* of a graph is an assignment of colors to the vertices such that no two adjacent vertices receive the same color. Colorings can be used to model many scheduling or partitioning problems. The *chromatic number* of a graph G is the minimum number of colors needed for a proper coloring and is denoted $\chi(G)$.

2.1.1 4-critical graphs with few edges

For an integer k, a graph is *k-critical* if it has chromatic number k but every proper subgraph has a proper coloring with $k-1$ colors. The only 2-critical graph is K_2, and the only 3-critical graphs are odd cycles. Chen, Erdős, Gyárfás, and Schelp [4] conjectured that 4-critical n-vertex graphs that are obtained from bipartite graphs by adding three edges must have significantly more than $5n/3$ edges, and suggested that perhaps they should have at least asymptotically $2n$ edges. They provided infinite families of such graphs with $2n-3$ edges.

Theorem 2.1 (Kostochka, Reiniger [10]). *A 4-critical n-vertex graph that is bipartite plus 3 edges has at least $2n-3$ edges.*

Our proof involves a potential function and the connection between orientations and colorings of graphs *from lists*. List coloring is a further restriction on coloring problems. Instead of having a universal set of colors available, now each vertex gets its own list of available colors. A graph is called k-choosable if there is a proper coloring for any assignment of lists of size k to the vertices. The choosability of a graph G, denoted $\mathrm{ch}(G)$, is the minimum k such that the graph is k-choosable. A graph's choosability is always at least its chromatic number, and can be arbitrarily larger.

The proof of Theorem 2.1 could have been shortened if certain sparse bipartite graphs were 3-choosable, but we have found counterexamples. Extending these examples leads to the next section.

2.1.2 Sparsity, girth, and coloring

With Alon, Kostochka, West, and Zhu [1], we constructed examples that show that the choosability bound coming from orientations is sharp. We have furthermore found examples without short cycles (i.e., arbitrarily large *girth*). Our method also provides examples of sparse graphs (and hypergraphs) with large chromatic number and large girth; Erdős first proved–probabilistically–that such graphs exist, and our construction improves on some subsequent explicit constructions.

Both of these constructions are based on a new structure: an *augmented tree* is a rooted tree with additional edges from each leaf to its ancestors. We prove that augmented trees exist with a variety of additional properties.

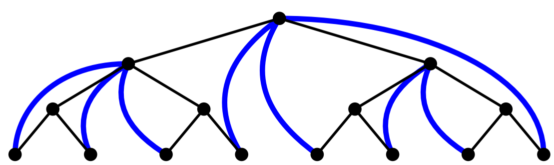

Figure 1: A 1-augmented binary tree.

While these constructions are quite pleasing, augmented trees must be very large if they have large girth. We are currently considering ways to decrease the size of the examples by relaxing the restrictions on the augmented trees.

2.1.3 List-coloring powers of graphs

Classes of graphs whose choosability is equal to their chromatic number–called *chromatic-choosable*–are particularly interesting and have long been studied.

The *mth power* of a graph G is the graph G^m with the same vertex set as G and with an edge between vertices u and v if and only if the distance from u to v in G is at most m. Motivated by total coloring of graphs, Kostochka and Woodall conjectured that squares of graphs are chromatic-choosable. Kim and Park [8] disproved this conjecture, and Xuding Zhu asked whether there exists an m such that mth powers are chromatic-choosable.

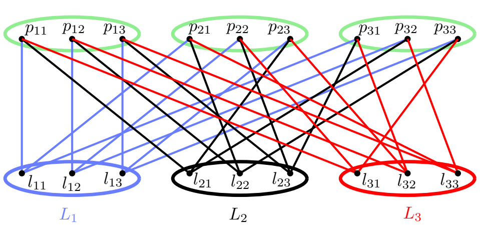

Figure 2: A graph whose cube is complete multipartite on the shown parts; larger such graphs prove Theorem 2.2 in the case $m = 3$.

Using modified affine planes, we provided infinite families of mth power graphs whose choosability greatly exceeds their chromatic number.

Theorem 2.2 (Kosar, Petrickova, Reiniger, Yeager [9])**.** *There is a $c > 0$ such that for each $m \geq 1$, there is an infinite family of graphs G with $\chi(G^m) \to \infty$ and*

$$\mathrm{ch}(G^m) \geq c\,\chi(G^m)\log\chi(G^m).$$

However, the choosability of proper graph powers is bounded by a polynomial function of their chromatic number:

Theorem 2.3 (KPRY [9])**.** *For every $m \geq 2$ and every G, $\mathrm{ch}(G^m) < \chi(G^m)^3$.*

I would like to improve either the lower or upper bound on choosability of graph powers in terms of their chromatic number; already in the case of squares this is interesting. I believe some improvement could be gained from a deeper understanding of the upper bound argument.

2.1.4 Dynamic colorings of graphs on surfaces

Dynamic coloring is another further restriction to coloring problems. An *r-dynamic coloring* is a proper coloring in which each vertex v has at least $\min\{r, d(v)\}$ different colors on its neighbors. The r-dynamic chromatic number of a graph G is denoted $\chi_r(G)$, and the r-dynamic choosability is denoted $\mathrm{ch}_r(G)$. Dynamic coloring models resource allocation in which each vertex should have many resources at its disposal (at adjacent vertices). We have

$$\chi(G) = \chi_1(G) \le \chi_2(G) \le \cdots \le \chi_{\Delta(G)}(G) = \chi(G^2),$$

so dynamic coloring also fills in information between the chromatic number of a graph and the chromatic number of its square.

A graph is *planar* (resp. *toroidal*) if it can be drawn in the plane (resp. torus) without crossing edges. The chromatic number of squares of planar graphs is well-studied; it can be arbitrarily large, but is bounded by a function of maximum degree. The 2-dynamic chromatic number of planar graphs is at most 5, and the only planar graph needing 5 colors is C_5 ([7]).

Theorem 2.4 (Loeb, Mahoney, Reiniger, Wise [11]). *For any planar or toroidal graph G, $\chi_3(G) \le \mathrm{ch}_3(G) \le 10$.*

We prove this theorem using discharging techniques. The Petersen graph (which is toroidal) achieves equality. Our best example of a planar graph with large 3-dynamic chromatic number is obtained from K_4 by subdividing the three edges incident to one vertex: it has 3-dynamic chromatic number 7.

We also proved bounds for r-dynamic coloring graphs of larger genus, and for larger r.

Theorem 2.5 (LMRW [11]). *Let G have genus g. If $g \le 2$ and $r \ge 2g + 11$, then $\mathrm{ch}_r(G) \le (g + 5)(r + 1) + 3$. If $g \ge 3$ and $r \ge 4g + 5$, then $\mathrm{ch}_r(G) \le (2g + 2)(r + 1) + 3$.*

The proof of this theorem is also based on discharging, but uses only reducible configurations involving "light edges," i.e. for which the sum of the degrees of their endpoints is small.

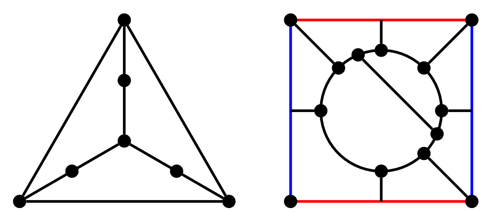

Figure 3: On the left, a planar graph G with $\chi_3(G) = 7$. On the right, a toroidal embedding of the Petersen graph P, which has $\chi_3(P) = 10$.

2.2 Degree sequences

The *degree sequence* of a graph is the list of its vertices' degrees, usually taken in nonincreasing order. The question of when a sequence of numbers is the degree sequence of some simple graph (called a *realization* of the sequence) is quite well understood.

2.2.1 Uniform hypergraphs

The same question for uniform hypergraphs is still open, and may be NP-hard. (A *k-uniform hypergraph* consists of a set of vertices and a set of edges, where each edge is a set of k vertices.) In a summer research group, we proved several sufficient conditions for a sequence to be the degree sequence of a k-uniform hypergraph. Using an involved induction argument together with poset considerations, we have the following.

Theorem 2.6 (Behrens, Erbes, Ferrara, Hartke, Reiniger, Spinoza, Tomlinson [2]). *Let π be a nonincreasing sequence with maximum term Δ, and let p be the minimum integer such that $\Delta \leq \binom{p-1}{k-1}$. If the sum of π is divisible by k and is at least $(\Delta - 1)p + 1$, then π is the degree sequence of some k-uniform hypergraph.*

We studied analogues of the 2-switch for ordinary graphs in the k-uniform case. An *i-exchange* in a uniform hypergraph is an operation that deletes i edges and adds i edges in such a way that vertex degrees are preserved. If an i-exchange does not result in any multiple edges, call it an *i-switch.*

Theorem 2.7 (BEFHRST [2]). *Any two realizations of a degree sequence of a k-uniform hypergraph can be transformed into one another by a sequence of 2-exchanges.*

We also proved that "2-exchanges" cannot be replaced by "2-switches" in the above theorem. In fact, we proved that i-switches do not suffice for any $i < k$:

Theorem 2.8 (BEFHRST [2]). *For every $k \geq 3$, there exists a degree sequence with two k-uniform realizations, neither of which admits an i-switch for any $i < k$.*

Question 2.9. *Do k-switches suffice to transform from any realization to any other?*

This question may be important to the resolution of the general question of when a sequence is the degree sequence of a k-uniform hypergraph. I have also lately been considering linear programming approaches to this main question.

2.2.2 Potential Ramsey numbers

The classical Ramsey number of graphs H_1 and H_2 is the minimum integer N such that every red/blue-coloring of the edges of K_N has either a red copy of H_1 or a blue copy of H_2. The *potential Ramsey number* of H_1 and H_2, denoted $r_{pot}(H_1, H_2)$, is the minimum integer N such that every degree sequence π of length N has some realization $G(\pi)$ such that either G contains a copy of H_1 or its complement $\overline{G}$ contains a copy of H_2. Analogous to the classical result of Chvátal [5], we prove the following.

Theorem 2.10 (Cox, Ferrara, Martin, Reiniger [6]). *If T_t is a tree with $|V(T_t)| = t$, and $t \geq 7(s-2)$, then $r_{pot}(K_s, T_t) = t + s - 2$.*

In order to prove this theorem, we specialize the packing result of Sauer and Spencer [13] to the case when one graph is a forest. Two graphs on n vertices *pack* if they can be found as edge-disjoint subgraphs of K_n.

Theorem 2.11 (CFMR [6]). *Let F be a forest, let $\ell(F)$ denote the number of vertices of F with degree 1, and let $\operatorname{comp}(F)$ denote the number of components of F that contain at least one edge. If $3\Delta(G) + \ell(F) - 2\operatorname{comp}(F) < n$, then F and G pack.*

3 Partially ordered sets

A *partially ordered set*, or *poset*, is a set with a reflexive, antisymmetric, transitive relation. The *dimension* of a poset P is the smallest k such that P is a subposet of $\mathbb{R}^k$ (in $\mathbb{R}^k$, $p \leq q$ if each coordinate of $q - p$ is nonnegative). In many ways, the dimension of posets is analogous to the chromatic number of graphs.

Dorais asked for the largest number $F_d(n)$ such that every n-element poset has a subposet of dimension at most d with at least $F_d(n)$ elements. Goodwillie showed that $F_d(n) \geq \sqrt{dn}$ using Dilworth's Theorem. We provided a sublinear upper bound, whose exponent depends on d. The following is our theorem for $d = 2$.

Theorem 3.1 (Reiniger, Yeager [12])**.** $F_2(n) \leq n^{0.8295}$.

There are similar questions that have been asked for graphs and chromatic number. In general one may ask, "given a large structure in which every small substructure is simple, how complicated can the large structure be?". The terms "small," "large," "simple," and "complicated" depend on the context. I am in particular curious about the following question.

Question 3.2 (RY [12])**.** *Given a poset P with n elements such that every m-element subposet has dimension at most d, how large can the dimension of P be?*

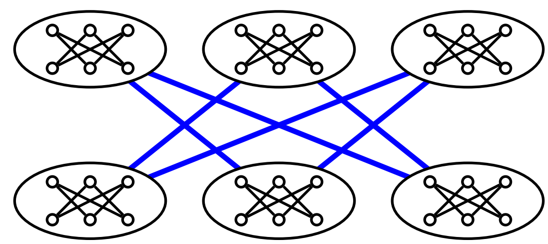

Figure 4: A schematic of the type of posets used to prove Theorem 3.1.

4 Competitive optimization

4.1 Game saturation number

In [3], we introduced the parameter of *game saturation number*. Given a host graph G and a forbidden family of graphs $\mathcal{F}$, consider the following game. Start with the edgeless subgraph H of G. Two players, Max and Min, alternate turns (starting with Max) adding an edge from $E(G) - E(H)$ into H, with the restriction that H never contains a copy of a graph of $\mathcal{F}$. The game ends when no move is possible. Max's goal is to maximize the length of the game (and hence the size of H), and Min's goal is to minimize it. The *graph saturation number* of $\mathcal{F}$ relative to G, denoted $\mathrm{sat}_g(\mathcal{F}; G)$, is the length of the game under optimal play by both players.

We computed the game saturation numbers for several forbidden families. Let $\mathcal{O}$ denote the set of all odd cycles, let $\mathcal{T}_n$ denote the set of all n-vertex trees, and let $\mathrm{sat}_g(F, G) = \mathrm{sat}_g(\{F\}, G)$. Then among our results are the following.

Theorem 4.1 (Carraher, Kinnersley, Reiniger, West [3])**.** $\mathrm{sat}_g(\mathcal{O}; K_{2k}) = k^2$.

$\mathrm{sat}_g(\mathcal{T}_n; K_n) = \binom{n-2}{2} + 1$ *for* $n \geq 6$.

$\mathrm{sat}_g(K_{1,3}; K_n) = 2\lfloor n/2 \rfloor$ *for* $n \geq 8$.

$\left|\mathrm{sat}_g(P_4; K_n) - \frac{4n-1}{5}\right| \leq 1$.

$$\text{With } m \geq n,\ \mathrm{sat}_g(P_4; K_{m,n}) = \begin{cases} n & \text{if } n \text{ is even,} \\ m & \text{if } n \text{ is odd and } m \text{ is even,} \\ m + \lfloor n/2 \rfloor & \text{if } mn \text{ is odd.} \end{cases}$$

$\mathrm{sat}_g(C_4; K_{n,n}) \geq \frac{1}{10.4} n^{13/12} - O(n^{35/36})$.

4.2 Pursuit-evasion games

The game of cops and robber is played on a graph, the cops and the robber taking turns moving to adjacent vertices. The cops win if they eventually capture the robber, and the robber wins if he can avoid capture indefinitely.

I have recently been considering variants of this game, including restricting the movements of either or both teams, or introducing randomness to the cops' movements.

References

[1] N. Alon, A. Kostochka, B. Reiniger, D. West, and X. Zhu. Coloring, sparseness, and girth. *Israel J. Math.* To appear.

[2] S. Behrens, C. Erbes, M. Ferrara, S. G. Hartke, B. Reiniger, H. Spinoza, and C. Tomlinson. New results on degree sequences of uniform hypergraphs. *Electron. J. Combin.*, 20(4):1–18 Paper 14, 2013.

[3] J. Carraher, W. Kinnersley, B. Reiniger, and D. B. West. The game saturation number of a graph. Submitted.

[4] G. Chen, P. Erdős, A. Gyárfás, and R. H. Schelp. A class of edge critical 4-chromatic graphs. *Graphs Combin.*, 13(2):139–146, 1997.

[5] V. Chvátal. Tree-complete graph Ramsey numbers. *J. Graph Theory*, 1(1):93, 1977.

[6] C. Cox, M. Ferrara, R. Martin, and B. Reiniger. Chvátal-type results for degree sequence Ramsey numbers. Submitted.

[7] S.-J. Kim, S. J. Lee, and W.-J. Park. Dynamic coloring and list dynamic coloring of planar graphs. *Discrete Appl. Math.*, 161(13-14):2207–2212, 2013.

[8] S.-J. Kim and B. Park. Counterexamples to the list square conjecture. *J. Graph Theory*, 78:239–247, 2015.

[9] N. Kosar, S. Petrickova, B. Reiniger, and E. Yeager. A note on list-coloring powers of graphs. *Discrete Math.*, 332:10–14, 2014.

[10] A. V. Kostochka and B. Reiniger. The minimum number of edges in a 4-critical graph that is bipartite plus 3 edges. *European J. Combin.*, 46:89–94, 2015.

[11] S. Loeb, T. Mahoney, B. Reiniger, and J. Wise. Dynamic coloring parameters for graphs with given genus. In preparation.

[12] B. Reiniger and E. Yeager. Large subposets with small dimension. *Order.* To appear.

[13] N. Sauer and J. Spencer. Edge disjoint placement of graphs. *J. Combin. Theory Ser. B*, 25(3):295–302, 1978.